

A Continuous-Time Markov Chain Approach for Availability Analysis of a Repairable Engineering System

Dr. M. Gayathri

Associate Professor

Government First Grade College, Varthur, Bangalore, Karnataka, India

Abstract:

Availability analysis is an important aspect of reliability engineering as it provides a quantitative measure of the operational performance of repairable engineering systems. This paper presents a Continuous-Time Markov Chain (CTMC) approach for evaluating the availability of a repairable engineering system consisting of eight interacting subsystems. The system behavior is represented by a 32-state transition model, which incorporates various operating, degraded, and failed conditions. Based on the state transition diagram, the corresponding Kolmogorov forward differential equations are formulated to obtain the state probabilities of the system. These state probabilities are then used to derive the system availability expression. The proposed model provides a comprehensive representation of the stochastic behavior of the system under failure and repair conditions. The analytical results demonstrate the applicability of the proposed approach for evaluating system availability and provide useful information for maintenance planning and improving the operational performance of complex repairable engineering systems.

Keywords: Continuous-Time Markov Chain, Repairable engineering system, Availability Analysis, Kolmogorov Forward Differential Equations, Markov Model.

1. Introduction

Availability analysis has become an important area of research in reliability engineering because uninterrupted operation of engineering systems is essential for improving productivity, reducing maintenance costs, and ensuring system safety. Modern engineering systems consist of several interdependent subsystems whose failures may significantly affect the overall performance of the system. Therefore, the development of appropriate mathematical models for evaluating system availability has received considerable attention from researchers.

Among the various stochastic techniques available for analyzing repairable systems, the Continuous-Time Markov Chain (CTMC) approach has proved to be one of the most effective methods. Markov models provide a systematic framework for describing the random transitions of system among different operating, degraded, and failed states. By assuming exponentially distributed failure and repair times, the behavior of a repairable system can be represented through a finite number of states, and the corresponding Kolmogorov forward differential equations can be formulated to determine the state probabilities.

Several researchers have successfully applied Markov modelling techniques to evaluate the performance of repairable systems in manufacturing industries, communication systems, power plants, and other

engineering applications. These studies have demonstrated that Markov models are capable of important performance measures and serve as an effective tool for maintenance planning and operational decision-making. However, engineering systems become increasingly complex, there is a need for more comprehensive state-space models capable of representing additional operating and failure conditions.

In the present work, a repairable engineering system consisting of eight interacting subsystems is considered. The system behavior is represented by a 32-state Continuous-Time Markov Chain, which incorporate normal operating, degraded, and failed conditions of the system. Based on the developed state transition diagram, the corresponding Kolmogorov forward differential equations are formulated to determine the state probabilities. These probabilities are subsequently used to obtain the availability expression of the system and to evaluate its operational performance under stochastic failure and repair conditions.

2. Literature Review

Continuous-Time Markov Chain (CTMC) models have been widely employed for analyzing the availability and performance of repairable engineering systems because of their ability to represent stochastic transitions between operational and failed states. The Markov approach provides an effective mathematical framework for modelling failure and repair processes and has been successfully applied to various industrial and engineering systems.

Brown (1984) established the theoretical foundation for analyzing repairable systems using stochastic processes and demonstrated the usefulness of Markov models in evaluating system performance measures. Islamov (1994) developed a generalized Markov reliability model for multiple repairable systems operating under different maintenance strategies. The study demonstrated that large systems of Kolmogorov forward differential equations could be effectively analyzed to evaluate system performance measures, making the approach suitable for complex repairable systems.

Rubino and Sericola (1995) proposed Markov-based dependability model for repairable systems and introduced an efficient computational technique based on randomization. Their work showed that Continuous-Time Markov Chains provide an effective tool for evaluating system availability and dependability.

Deepika Garg, Kuldeep Kumar, and Jai Singh (2010) presented the availability analysis of a cattle feed plant using a matrix method based on the Markov birth-death process. The plant consisted of seven major subsystems, of which two operated through standby units with perfect switching devices. The state probabilities were obtained by solving the governing differential equation, and the variation of system availability with time was analyzed. This work provides the methodological foundation for the present study.

Yusuf (2012) analyzed a repairable 3-out-of-4 redundant system with preventive maintenance using the Chapman-Kolmogorov approach. Explicit expressions for steady-state availability and profit function were obtained, and the influence of failure and repair rates on system performance was investigated.

Yusuf (2013) developed a stochastic model for a repairable warm standby system with dedicated repair facilities. Using Kolmogorov forward differential equations, the author evaluated important performance measures including availability, repair facility utilization, and mean system life, demonstrating the applicability of Markov modelling to repairable systems with standby redundancy.

Aggarwal, Kumar, Singh, and Garg (2015) developed a Markov model for the reliability analysis of the urea synthesis system of fertilizer plant. Their study showed that proper maintenance planning based on subsystem failure and repair characteristics can significantly improve long-run system availability and reliability.

Yu et al. (2018) proposed a Continuous-Time Markov model for multi-state repairable systems considering degradation and imperfect repair. Their work demonstrated that increasing the number of system states provides a more realistic representation of engineering systems and improves the accuracy of availability prediction.

Chatterjee and Sarkar (2020) developed a methodology for computing the limiting average availability of repairable systems with multiple spare units and repair facilities. Their discretization approach extended the applicability of availability analysis beyond conventional exponential assumptions and provided an effective computational alternative for practical engineering systems.

Yang and Wu (2021) investigated the availability and reliability of a standby repairable system incorporating imperfect switchovers and repairperson breakdowns. A finite Continuous-Time Markov Chain model was formulated, and the governing differential equations were solved to evaluate time-dependent availability under different operating conditions.

Recent studies have further extended Markov-based availability analysis to more realistic repairable systems. Juybari, Hamadani, and Ardakan (2024) investigated a repairable system with warm-standby components operating under environmental shocks using a Markovian framework and analyzed both transient and steady-state availability together with maintenance optimization. Oszczypala et al. (2024) proposed a Continuous-Time Markov Chain (CTMC) model for reliability analysis and redundancy optimization of repairable k-out-of-n systems with sensitivity analysis. Similarly, a CTMC based availability optimization framework integrating cold, warm, and hot standby redundancies was developed for repairable systems using a genetic algorithm in 2024.

These studies demonstrate the growing importance of CTMC-based modelling in analyzing complex repairable engineering systems and motivate the development of the present 32-state CTMC model for a repairable system with eight subsystems and reduced-capacity operating states

Although several studies have applied Markov modelling to repairable engineering systems, most existing models consider comparatively fewer operating and degraded states. Complex engineering systems often require a larger state-space to represent various operating conditions and repair transitions more accurately. Motivated by this observation, the present work develops a 32-state Continuous-Time Markov Chain model to evaluate the availability of a repairable engineering system. The proposed model provides a

detailed state-space representation and facilitates comprehensive availability analysis under stochastic failure and repair processes.

3. System Description

The bottling plant considered in this study consists of eight major subsystems, namely Bottle-Feeding Unit(A), Washing Unit(B), Filling Unit(C), Capping Unit(D), labeling Unit (E), Inspection unit (F), Packaging Unit (G), and Conveyor Unit (H). Initially, the bottle-feeding unit supplies empty bottle-feeding unit supplies empty bottles to the washing unit, where the bottles to the washing unit, where the bottles are cleaned before entering the filling unit. The filling unit dispenses the required quantity of liquid into the bottles, after which the capping unit seals them. The bottles then move to the labeling unit for labeling, followed by inspection to identify defective products. Finally, the accepted bottles are transferred to the packaging unit and conveyed to the storage area through the conveyor unit. The system configuration is represented by a 32-state Continuous-Time Markov Chain, which is used to evaluate the system availability.

The bottling plant consists of the following eight main subsystems:

1. Bottle Feeding Unit (A): Consists of one operating unit. The system becomes unavailable whenever this subsystem fails.
2. Washing Unit (B): Consists of two units, one operating and other in cold standby. The standby unit operates in a reduced-capacity state. Complete failure occurs when both units fail.
3. Filling Unit (C): Consists of one operating unit. Failure of this subsystem causes the system to become unavailable.
4. Capping Unit (D): Consists of one operating unit. The system becomes unavailable whenever this subsystem fails.
5. Labeling Unit (E): Consists of two units, one operating and other in cold standby. The standby unit operates in a reduced-capacity state, and complete failure occurs when both units fail.
6. Inspection Unit (F): Consists of one operating unit. Failure of this subsystem results in system unavailability.
7. Packaging Unit (G): Consists of one operating unit. The system becomes unavailable whenever this subsystem fails.
8. Conveyor Unit (H): Consists of one operating unit, Failure of this subsystem causes the entire system to become unavailable.

4. Assumptions:

1. The failure and repair times of all subsystems are exponentially distributed and are statistically independent of each other.
2. At any instant of time, not more than one subsystem fails.
3. A repaired subsystem is restored to an as-good-as-new condition.
4. The subsystems B and E operate through reduced-capacity states before complete failure.
5. The standby units associated with subsystems B and E are cold standby unit and become operative only after the failure of the corresponding operating units.
6. The switch-over devices from the operating unit to the standby unit are assumed to be perfect and instantaneous.

7. The failure and repair rates are constant throughout the operation of the system.

5. Notations:

S. No.	Symbols	Description
1	A, B, C, D, E, F, G, H	The Normal operating states of the respective subsystems
2	(B), (E)	Denote the reduced-capacity states of subsystem B and E, respectively
3	a, b, c, d, e, f, g, h	Denote the failed states of subsystems A, B, C, D, E, F, G, and H respectively
4	λ_i	Failure rate of the corresponding subsystem, $i=1,2,3,4,5,6,7,8,9,10$. The failure rates corresponding to the reduced-capacity states of B and E are represented by the appropriate λ_i values.
5	μ_i	Repair rate of the corresponding subsystem, $i=1,2,3,4,5,6,7,8,9,10$. The repair rates corresponding to the reduced-capacity states of B and E are represented by the appropriate μ_i values.
6	$P_i(t)$	Probability that the system is in state i at time t
7	$P'_i(t)$	Derivative of the probability function $P_i(t)$
8	A	Transition Matrix
9	I	Identity Matrix
10	P (0)	Initial probability vector
11	Av(t)	Availability

6. System Transition diagram:

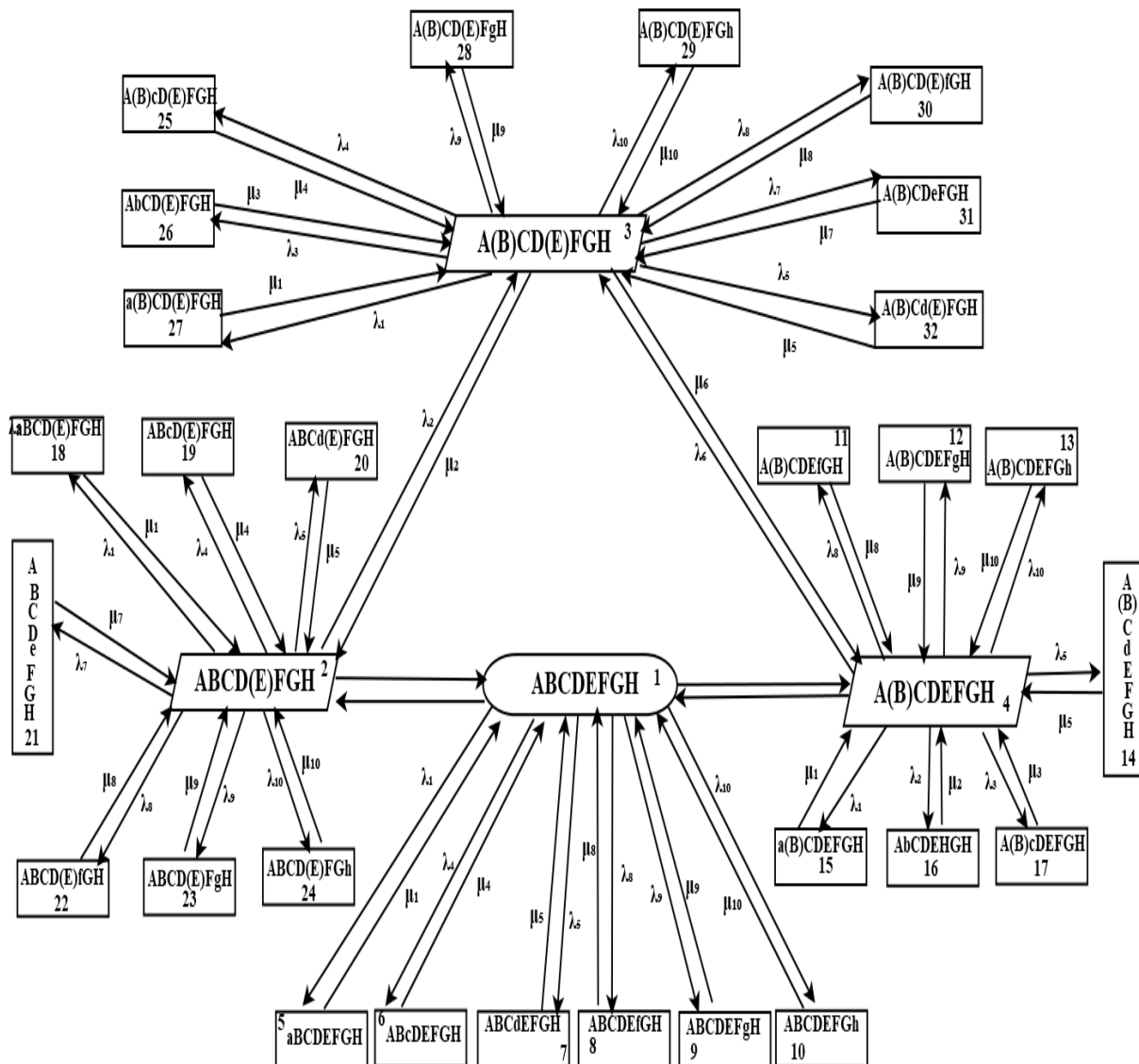


Fig. 1: System Transition Diagram

7. Mathematical Modelling:

From system transition diagram we can write the following differential equations

$$P_1^1(t) = -(\lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_8 + \lambda_9 + \lambda_{10})P_1(t) + \mu_1 P_5(t) + \mu_2 P_4(t) + \mu_4 P_6(t) + \mu_5 P_7(t) + \mu_6 P_2(t) + \mu_8 P_8(t) + \mu_9 P_9(t) + \mu_{10} P_{10}(t) \quad (1)$$

$$P_2^1(t) = -(\lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 + \lambda_7 + \lambda_8 + \lambda_9 + \lambda_{10} + \mu_6)P_2(t) + \mu_1 P_{18}(t) + \mu_2 P_3(t) + \mu_4 P_{19}(t) + \mu_5 P_{20}(t) + \mu_6 P_1(t) + \mu_7 P_{21}(t) + \mu_8 P_{22}(t) + \mu_9 P_{23}(t) + \mu_{10} P_{24}(t) \quad (2)$$

$$P_3^1(t) = -(\lambda_1 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_7 + \lambda_8 + \lambda_9 + \lambda_{10} + \mu_2 + \mu_6)P_3(t) + \mu_1 P_{27}(t) + \mu_2 P_2(t) + \mu_3 P_{26}(t) + \mu_4 P_{25}(t) + \mu_5 P_{32}(t) + \mu_6 P_4(t) + \mu_7 P_{31}(t) + \mu_8 P_{30}(t) + \mu_9 P_{28}(t) + \mu_{10} P_{29}(t) \quad (3)$$

$$P_4^1(t) = -(\lambda_1 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_8 + \lambda_9 + \lambda_{10} + \mu_2)P_4(t) + \mu_1 P_{15}(t) + \mu_3 P_{16}(t) + \mu_4 P_{17}(t) + \mu_5 P_{14}(t) + \mu_6 P_3(t) + \mu_7 P_{21}(t) + \mu_8 P_{11}(t) + \mu_9 P_{12}(t) + \mu_{10} P_{13}(t) \quad (4)$$

$$\text{Let } a_1 = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_8 + \lambda_9 + \lambda_{10}$$

$$a_2 = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 + \lambda_7 + \lambda_8 + \lambda_9 + \lambda_{10} + \mu_6$$

$$a_3 = \lambda_1 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_7 + \lambda_8 + \lambda_9 + \lambda_{10} + \mu_2 + \mu_6$$

$$a_4 = \lambda_1 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_8 + \lambda_9 + \lambda_{10} + \mu_2$$

$$P_i^1(t) + \mu_j P_i(t) = \lambda_j P_1(t) \quad (5)$$

where $i = 5, 6, 7, 8, 9, 10$ and $j = 1, 4, 5, 8, 9, 10$

$$P_i^1(t) + \mu_j P_i(t) = \lambda_j P_4(t) \quad (6)$$

where $i = 11, 12, 13, 14, 15, 16, 17$ and $j = 8, 9, 10, 5, 1, 3, 4$

$$P_i^1(t) + \mu_j P_i(t) = \lambda_j P_2(t) \quad (7)$$

where $i = 18, 19, 20, 21, 22, 23, 24$ and $j = 1, 4, 5, 7, 8, 9, 10$

$$P_i^1(t) + \mu_j P_i(t) = \lambda_j P_3(t) \quad (8)$$

where $i = 25, 26, 27, 28, 29, 30, 31, 32$ and $j = 4, 3, 1, 9, 10, 8, 7, 5$

with initial conditions $P_1(0) = 1$, otherwise zero.

Let $P(k, t)$ denotes the transition probability of the event that the system is in state k at time t . Since the number of all the possible transition states of the complex system is 32. So, the system of differential difference equations for above equations may be written as,

$$(\bar{\theta}I - A)\bar{P}(k, t) = \bar{0}, \text{ where } \bar{\theta} = \frac{d}{dt}, \bar{0} \text{ is the null matrix, matrix } A \text{ is the matrix of coefficients of } P_i(t) \text{ 's}$$

in differential difference equation, $\bar{P}(k, t) = [P_1(t) P_2(t) \dots P_{32}(t)]^T$ and

Matrix $I_{32 \times 32}$ is the identity matrix.

$$A = [a_{ij}]_{32 \times 32} \text{ where } a_{ij} = \begin{cases} -\sum (\text{all rates leaving state } i), & i = j \\ \lambda_k \text{ or } \mu_k & \text{if a transition exists} \\ 0 & \text{otherwise} \end{cases}$$

$A(1,1) = a_1$	$A(2,1) = \lambda_6$	$A(3,2) = \lambda_2$	$A(4,1) = \lambda_2$	$A(5,1) = \lambda_1$	$A(11,4) = \lambda_8$
$A(1,2) = \mu_6$	$A(2,2) = a_2$	$A(3,3) = a_3$	$A(4,3) = \mu_6$	$A(5,5) = -\mu_1$	$A(11,11) = -\mu_8$
$A(1,4) = \mu_2$	$A(2,3) = \mu_2$	$A(3,4) = \lambda_6$	$A(4,4) = a_4$	$A(6,1) = \lambda_4$	$A(12,4) = \lambda_9$
$A(1,5) = \mu_1$	$A(2,18) = \mu_1$	$A(3,25) = \mu_4$	$A(4,11) = \mu_8$	$A(6,6) = -\mu_4$	$A(12,12) = -\mu_9$
$A(1,6) = \mu_4$	$A(2,19) = \mu_4$	$A(3,26) = \mu_3$	$A(4,12) = \mu_9$	$A(7,1) = \lambda_5$	$A(13,4) = \lambda_{10}$
$A(1,7) = \mu_5$	$A(2,20) = \mu_5$	$A(3,27) = \mu_1$	$A(4,13) = \mu_{10}$	$A(7,7) = -\mu_5$	$A(13,13) = -\mu_{10}$
$A(1,8) = \mu_8$	$A(2,21) = \mu_7$	$A(3,28) = \mu_9$	$A(4,14) = \mu_5$	$A(8,1) = \lambda_8$	$A(14,4) = \lambda_5$

$A(1,9) = \mu_9$	$A(2,22) = \mu_8$	$A(3,29) = \mu_{10}$	$A(4,15) = \mu_1$	$A(8,8) = -\mu_8$	$A(14,14) = -\mu_5$
$A(1,10) = \mu_{10}$	$A(2,23) = \mu_9$	$A(3,30) = \mu_8$	$A(4,16) = \mu_3$	$A(9,1) = \lambda_9$	$A(15,4) = \lambda_1$
	$A(2,24) = \mu_{10}$	$A(3,31) = \mu_7$	$A(4,17) = \mu_4$	$A(9,9) = -\mu_9$	$A(15,15) = -\mu_1$
		$A(3,32) = \mu_5$		$A(10,1) = \lambda_{10}$	$A(16,4) = \lambda_3$
				$A(10,10) = -\mu_{10}$	$A(16,16) = -\mu_3$

$A(17,4) = \lambda_4$	$A(22,22) = -\mu_8$	$A(28,3) = \lambda_9$
$A(17,17) = -\mu_4$	$A(23,2) = \lambda_9$	$A(28,28) = -\mu_9$
$A(18,2) = \lambda_1$	$A(23,23) = -\mu_9$	$A(29,3) = \lambda_{10}$
$A(18,18) = -\mu_1$	$A(24,2) = \lambda_{10}$	$A(29,29) = -\mu_{10}$
$A(19,2) = \lambda_4$	$A(24,24) = -\mu_{10}$	$A(30,3) = \lambda_8$
$A(19,19) = -\mu_4$	$A(25,3) = \lambda_4$	$A(30,30) = -\mu_8$
$A(20,2) = \lambda_5$	$A(25,25) = -\mu_4$	$A(31,3) = \lambda_7$
$A(20,20) = -\mu_5$	$A(26,3) = \lambda_3$	$A(31,31) = -\mu_7$
$A(21,2) = \lambda_7$	$A(26,26) = -\mu_3$	$A(32,3) = \lambda_5$
$A(21,21) = -\mu_7$	$A(27,3) = \lambda_1$	$A(32,32) = -\mu_5$
$A(22,2) = \lambda_8$	$A(27,27) = -\mu_1$	

Transition state availability of different stages are:

$$P(1,t) = 1 + a_{11}t + b_{11} \frac{t^2}{2!} + \dots$$

$$P(2,t) = a_{21}t + b_{21} \frac{t^2}{2!} + \dots$$

.....

$$P(i,t) = a_{i1}t + b_{i1} \frac{t^2}{2!} + \dots$$

Here $AP(0) = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \\ \vdots \\ \vdots \\ \vdots \end{bmatrix}$, where $a_{11} = A(1,1), a_{21} = A(2,1), a_{31} = A(3,1), a_{41} = A(4,1)$

$$A^2P(0) = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \\ b_{41} \\ \vdots \\ \vdots \\ \vdots \end{bmatrix}, A^3P(0) = \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \\ c_{41} \\ \vdots \\ \vdots \\ \vdots \end{bmatrix}, A^4P(0) = \begin{bmatrix} d_{11} \\ d_{21} \\ d_{31} \\ d_{41} \\ \vdots \\ \vdots \\ \vdots \end{bmatrix}, \dots \text{ where } AP(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \end{bmatrix}_{32 \times 1}$$

Since $P(1,t), P(2,t), P(3,t), P(4,t)$ are the only working states of a system, so

$$\begin{aligned} AV(t) &= P(1,t) + P(2,t) + P(3,t) + P(4,t) \\ &= 1 + (a_{11} + a_{21} + a_{31} + a_{41})t + (b_{11} + b_{21} + b_{31} + b_{41})\frac{t^2}{2!} + \dots \\ &= 1 - 0.0140000t + 0.000243500t^2 + \dots \end{aligned}$$

8. Numerical results:

We calculate the availability of the system at different time t, we get the following table.

S. No.	Time t	AV(t)
1.	10	0.8820
2.	20	0.7992
3.	30	0.7408
4.	40	0.6998
5.	50	0.6710
6.	60	0.6508
7.	70	0.6366
8.	80	0.6266
9.	90	0.6195
10.	100	0.6145

Table 1: Variation of Availability with Time

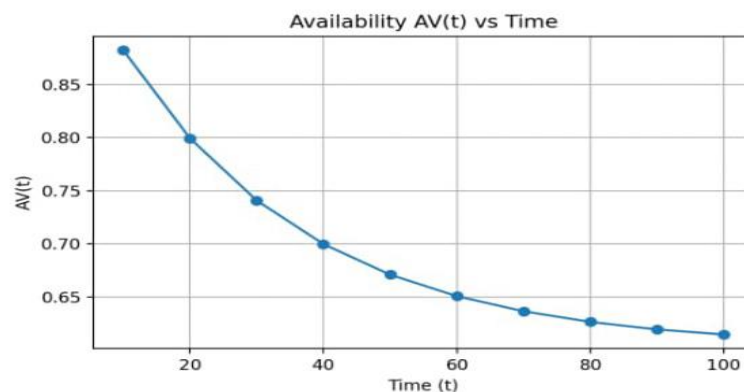


Fig. 2

9. Conclusion:

This study presents a Continuous-Time Markov Chain (CTMC)- based availability model for eight-subsystem for an eight-subsystem Bottling plant. The developed model effectively represents the stochastic failure and repair behaviour of the system provide a mathematical frame work for evaluating its operational performance.

The numerical results indicate that the system availability decreases gradually with increasing operating time, as illustrated in Fig.2, Initially, the system exhibits high availability; however, continuous operation and the accumulation of component failures result in a steady decline in availability. The smooth decreasing trend confirms the influence of failure rates on long-term system performance and highlights the importance of timely maintenance and repair activities.

The proposed model can assist plant managers and maintenance engineers in predicting system performance, scheduling preventive maintenance, minimizing downtime, and improving overall plant reliability. The methodology is general and can be extended to other complex industrial systems with multiple repairable subsystems. Future work may incorporate preventive maintenance policies, economic analysis, fuzzy reliability, or artificial intelligence techniques to further enhance the accuracy and practical applicability of the model.

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