

Article

Continuous-Time Markov Chain Modelling for Service Life Prediction of Building Elements

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Abstract

A continuous-time Markov chain framework is developed for service life prediction of building assets, and three formulations are compared: a homogeneous generator, a time-varying generator, and a fractional model. The framework delivers survival, density of absorption time, hazard, and mean time to absorption. For the homogeneous case, state trajectories are computed using matrix exponentials. The time-varying case is solved both by local exponential propagation on a time grid and by direct integration of the Kolmogorov equation. The fractional case is implemented in two independent ways, via a truncated series expansion and via an in-house routine for the Mittag-Leffler function, which also allows the direct evaluation of survival and hazard from the standard fractional relations while avoiding singular behaviour at the origin. This study shows that non-homogeneous rates accelerate deterioration relative to the homogeneous benchmark, whereas fractional dynamics reproduce early-time acceleration followed by a slow decline of the hazard, which is consistent with heavy-tailed survival and longer effective service life. The two fractional solvers provide mutually consistent outputs, which supports the numerical robustness of the approach. The framework is readily applicable to sparse inspection data and short observation windows and provides a transparent basis for comparing modelling assumptions that affect life cycle forecasts used in asset management and maintenance planning.

Keywords: continuous-time Markov chains; service life prediction; building deterioration modelling; time-varying transition rates; fractional calculus; Mittag-Leffler function



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1. Introduction

An accurate service life prediction has direct economic value. Replacement and major rehabilitation dominate life cycle costs. The timing of interventions depends on the shape of the survival and hazard curves, not only on average age [1]. Standards for life cycle costing require consistent service life inputs to compute the present value and compare alternatives, which links our outputs to budgeting and procurement practices [2,3]. Asset management frameworks require risk-based planning that combines the probability of failure and consequence of failure; thus, probabilistic forecasts support transparent decisions and audit trails [4]. In public infrastructure, life cycle cost analysis is a policy requirement, and agencies report savings when they schedule work using probabilistic models instead of fixed age rules [5]. Established costing methods in construction couple unit prices, discounting, and residual value to expected times spent in each condition

state, which means that Markov outputs can feed the economic model without additional assumptions [6].

Reliable service life prediction requires probabilistic tools. These tools connect routine inspections to quantities that engineers can act on, including survival, hazard, and mean time to absorption. Continuous-time Markov chains provide this link. They model deterioration as a stochastic flow between condition states on a real-time axis and do not force the analysis to jump only at inspection instants [7–9]. In this paper the methodology is organised so that a single dataset can be analysed under three complementary dynamics. The first variant uses a time-invariant generator and computes trajectories using matrix exponentials. The second one allows rates that change over time and propagates the system with local exponentials on a time grid. We also use an adaptive ordinary differential equation solver in MATLAB R2025a. The third introduces fractional-order dynamics to represent memory effects and heavy tails. We implement it in two independent ways: a truncated series and an in-house Mittag-Leffler routine that evaluates the required matrix function directly in MATLAB [10–13]. The output is presented as survival and hazard curves, together with the mean time to absorption. This makes comparisons between modelling assumptions clear for maintenance planning.

The study makes both practical and methodological contributions. We assemble a reproducible workflow that begins with inspection counts and ends with reliability outputs at any time of interest. We keep all numerical steps elementary and auditable. The time-varying formulation shows how changing rates modify the pace of ageing in a controlled manner. The fractional formulation shows early acceleration followed by a slow decline of the hazard, which is typical of long-memory deterioration. Throughout, figures report survival and hazard for the three variants. A compact table summarises the main indicators so that practitioners can see at a glance how modelling choices influence life cycle forecasts. Classic references on Markov modelling and reliability motivate this structure and terminology, and standard sources on fractional calculus justify the fractional extension.

The three models use the same condition states and the same inspection data, but they rely on different assumptions. The homogeneous model assumes constant rates in time and serves as the basic reference case. The time-varying model allows these rates to change with time, which is useful when deterioration is not stationary. The fractional model introduces memory effects and heavy-tailed behaviour, so it can represent deterioration paths that are not well described by standard Markov dynamics. In this sense, the time-varying model extends the homogeneous one, while the fractional model extends the classical framework toward non-memoryless behaviour. This paper continues our earlier study on Markov chain-based service life modelling of buildings and structural components. In that study the continuous-time idea was only outlined. Here, we develop the concept in full and turn it into a working reliability tool that complements the discrete-time and fuzzy formulations already presented [14].

The remainder of the paper is organised as follows. Section 2 introduces the continuous-time Markov framework and its homogeneous, time-varying, and fractional variants. Section 3 describes the empirical dataset and the condition-state classification. Section 4 presents the transition matrices and model identification procedure. Section 5 discusses the numerical results and their practical interpretation. Finally, Section 6 summarises the main conclusions and outlines directions for future work.

2. Continuous Markov Chain Modelling

A continuous-time Markov chain is used to represent deterioration as a stochastic flow across a finite set of condition states with one absorbing state that models failure [15–17]. The process is described by a row probability vector $\mathbf{p}(t) \in \mathbb{R}^{1 \times m}$ whose components are

non-negative and sum to one. The notation and orientation follow the continuous-time part of the uploaded article and remain consistent with its use of the forward equation and stationary behaviour:

$$\sum_{i=1}^m p_i(t) = 1 \quad (1)$$

The dynamics are defined by a generator matrix $\mathbf{Q} \in \mathbb{R}^{m \times m}$. Off-diagonal entries are transition rates and are non-negative. Each row sums to zero. These constraints guarantee mass conservation and probabilistic consistency:

$$Q_{ij} \geq 0 \text{ for } i \neq j, \sum_{j=1}^m Q_{ij} = 0 \quad (2)$$

The Kolmogorov forward equation governs the evolution of the probabilities [18]. The row form is adopted to match computational practice and the base paper:

$$\frac{d}{dt} \mathbf{p}(t) = \mathbf{p}(t) \mathbf{Q} \quad (3)$$

2.1. Constant Generator $\mathbf{Q}$

The time-homogeneous case assumes that the generator does not depend on time. The forward equation has a closed-form solution through the matrix exponential. This representation is exact and is the basis for the continuous curves used to compare with discrete inspection updates [19]:

$$\mathbf{p}(t) = \mathbf{p}(0) \exp(\mathbf{Q} t) \quad (4)$$

The stationary distribution $\boldsymbol{\pi} \in \mathbb{R}^{1 \times m}$ characterises the limit behaviour when it exists. It is the left eigenvector of $\mathbf{Q}$ for eigenvalue zero with unit sum. For chains with an absorbing state, the stationary vector concentrates all mass in that state:

$$\boldsymbol{\pi} \mathbf{Q} = \mathbf{0}, \sum_{i=1}^m \pi_i = 1 \quad (5)$$

In many applications a discrete transition matrix $\mathbf{P} \in \mathbb{R}^{m \times m}$ is available for a fixed inspection interval Δt . A common identification sets the generator by the matrix logarithm:

$$\mathbf{Q} = \frac{1}{\Delta t} \log \mathbf{P} \quad (6)$$

The logarithm of a matrix is defined by $\log \mathbf{P} = \mathbf{V} \log(\boldsymbol{\lambda}) \mathbf{V}^{-1}$ for a diagonalisable $\mathbf{P} = \mathbf{V} \boldsymbol{\lambda} \mathbf{V}^{-1}$, with $\log(\boldsymbol{\lambda})$ obtained by taking the complex logarithm of each eigenvalue. This construction uses the principal logarithm when all eigenvalues lie in the open right half-plane. In practice a numerically stable route applies the real Schur decomposition $\mathbf{P} = \mathbf{U} \mathbf{T} \mathbf{U}^T$, evaluates $\log \mathbf{T}$ by a block algorithm, and sets $\mathbf{P} = \mathbf{U} \log \mathbf{T} \mathbf{U}^T$ [20,21]. The approach is exact only when $\mathbf{P}$ is embeddable in a continuous-time chain. If embeddability fails, $\log \mathbf{P}$ may have negative off-diagonal entries or complex values, which violates the generator constraints.

2.2. Time-Varying Generator $\mathbf{Q}(t)$

Deterioration rates may change in time due to ageing, environment, or usage. A non-homogeneous Markov chain uses a generator that depends on time. The forward equation becomes

$$\frac{d}{dt}\mathbf{p}(t) = \mathbf{p}(t)\mathbf{Q}(t) \quad (7)$$

The exact solution can be written with a time-ordered exponential:

$$\mathbf{p}(t) = \mathbf{p}(0)\mathcal{T}\left[\exp\left(\int_0^t \mathbf{Q}(s) ds\right)\right] \quad (8)$$

where $\mathcal{T}[\cdot]$ denotes the time-ordering operator [22].

The direct evaluation of the time-ordering operator is rarely practical. A standard numerical strategy uses a short time step Δt , freezes the generator on each subinterval, and applies a local exponential [23,24]:

$$\mathbf{p}(t_{k+1}) = \mathbf{p}(t_k)\exp(\mathbf{Q}(t_k)\Delta t) \quad (9)$$

The local exponential is a first-order accurate Lie product integrator. It preserves the row sum equal to one and usually keeps entries non-negative when the step is sufficiently small. As an independent check, the differential system can also be integrated with an adaptive ODE method that realises the same row vector form. The base article uses the same forward equation and provides context for continuous simulation.

A generator can also be identified directly from structural assumptions. For example, a birth-type structure uses one positive rate per transient state [7]. Such constraints can be embedded in a least-squares fit that minimises Frobenius norm $\|\exp(\mathbf{Q}\Delta t) - \mathbf{P}\|_F$ subject to generator conditions $q_{ij} \geq 0$ ($i \neq j$) and $\sum_j q_{ij} = 0$. This route avoids unstable logarithms and respects probabilistic geometry.

2.3. Fractional Kolmogorov Dynamics

Some deterioration data show slower than exponential decay. Memory effects can be captured by replacing the first derivative with a Caputo derivative of order α in the open interval $(0, 1)$. The fractional forward equation reads as follows [11]:

$$D_t^\alpha \mathbf{p}(t) = \mathbf{p}(t)\mathbf{Q}, \quad 0 < \alpha < 1 \quad (10)$$

The solution involves the matrix Mittag-Leffler function. The scalar Mittag-Leffler function is defined by the power series:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad (11)$$

Here $\Gamma(\cdot)$ is the gamma function which extends the factorial to real and complex arguments, since $\Gamma(n+1) = n!$ for integer n . The matrix form applies this series to the matrix argument $\mathbf{Q}t^\alpha$. For the fractional Kolmogorov equation with $\beta = 1$ the probability vector is

$$\mathbf{p}(t) = \mathbf{p}(0)E_{\alpha,1}(\mathbf{Q}t^\alpha) \quad (12)$$

For small state spaces a direct series evaluation is effective. The truncated series

$$\mathbf{p}(t) \approx \mathbf{p}(0) \sum_{k=0}^K \frac{(\mathbf{Q}t^\alpha)^k}{\Gamma(\alpha k + 1)} \quad (13)$$

converges rapidly because the gamma function in the denominator grows superlinearly with k [25]. The growth of $\Gamma(\alpha k + 1)$ damps high-order terms and yields a stable computation. After truncation, small negative round-off errors can be clipped and the vector can be renormalised to enforce the simplex constraint $\sum_i p_i(t) = 1$.

The paper develops a continuous equation in the classical case. The fractional extension adds a memory kernel and produces heavier tails for the time to absorption, which is often useful for infrastructure deterioration.

The theoretical position of the three formulations should be stated clearly. The homogeneous and time-varying models are both continuous-time Markov chains. In the second case, the generator depends on time, but the process still preserves the Markov property. By contrast, the fractional model with the Caputo derivative is no longer a classical CTMC. The fractional operator introduces history dependence into the evolution law, so the future state probabilities depend on the past through a memory kernel. For this reason, the fractional formulation should be interpreted as a memory-based semi-Markov-type process.

This distinction is also reflected in the waiting-time structure. In a classical CTMC, the sojourn time in each state is exponential, which is equivalent to the memoryless property. In a classical semi-Markov process, the sojourn time may follow a general non-exponential distribution. In the fractional case studied here, the waiting-time behaviour is governed by Mittag-Leffler-type laws, which produce heavy tails and stronger memory effects than the standard Markov setting. Therefore, the proposed fractional model is best viewed as a fractional semi-Markov extension of the classical continuous-time Markov framework.

To make the waiting-time interpretation explicit, let T_i denote the sojourn time in transient state i , and let $\bar{F}_i(t)$ denote its tail function, that is, the probability that the process remains in state i longer than t . If $\lambda_i = -q_{ii} > 0$ is the total exit intensity from state i , then the fractional model gives

$$\bar{F}_i(t) = E_{\alpha,1}(-\lambda_i t^\alpha), \quad f_i(t) = -\frac{d}{dt}\bar{F}_i(t) = \lambda_i t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i t^\alpha) \quad (14)$$

Thus, the sojourn time follows the Mittag-Leffler distribution. This law is not imposed as an additional assumption, but follows directly from the fractional evolution equation. For $\alpha = 1$, the classical exponential case is recovered. For $0 < \alpha < 1$, the tail is algebraic rather than exponential:

$$\bar{F}_i(t) \sim \frac{1}{\lambda_i \Gamma(1-\alpha)} t^{-\alpha}, \quad t \rightarrow \infty \quad (15)$$

which explains the heavy-tailed character of the fractional model.

In contrast, phase-type distributions are constructed from finitely many exponential phases and therefore remain light-tailed [26]. The fractional waiting-time law also breaks the constant-hazard structure of the exponential case, which at the multi-state level may lead to the non-monotone hazard curves discussed later.

3. Service Life Planning Application

For the practical implementation of the proposed model, empirical data reflecting the deterioration of buildings were collected from a real investment consisting of two hotels, one 10-storey and one 6-storey, with underground two-level parking.

The 10-storey hotel and 6-storey hotel were designed as separate functional buildings situated above a two-storey underground car park. Tables 1 and 2 show the key technical parameters of the hotels and underground parking.

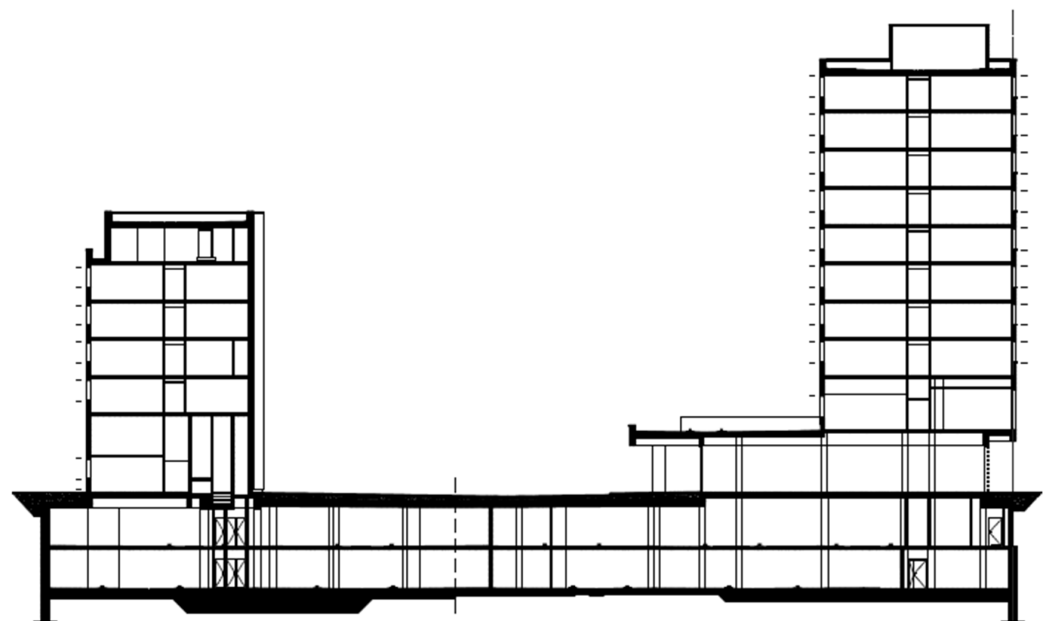
Table 1. Key technical parameters of hotels.

Hotels	6-Storey Hotel	10-Storey Hotel
Total area	3,911,125 m ²	589,002 m ²
	9,801,145 m ²	
Usable area	316,005 m ²	474,897 m ²
	790,902 m ²	
Cubature	142,899 m ³	1,652,237 m ³
	3,081,227 m ³	
No. of rooms	163	170

Table 2. Key technical parameters of underground parking.

Underground Parking	
Total area	65,078 m ²
Usable area	579,385 m ²
Cubature	2,278,982 m ³
No. of storeys	2

Figure 1 shows the cross-section of two hotels—one 10- and the other one 6-storey—with underground two-level parking. Figure 2 shows the ground floor plan of the hotels.

**Figure 1.** Cross-section: two hotels—one 10- and the other one 6-storey—with underground 2-level parking.

The dataset is based on systematic inspections and expert assessments conducted at five-year intervals. Each structural and non-structural component was assigned to a discrete condition state, ranging from full serviceability to advanced damage.

A four-level classification scheme was applied to describe the degree of wear:

- State 1: “Very good”, corresponding to element wear of 0–15%.
- State 2: “Medium”, corresponding to element wear of 16–30%.
- State 3: “Satisfactory”, corresponding to element wear of 31–50%.
- State 4: “Poor”, corresponding to element wear of 51–70%.

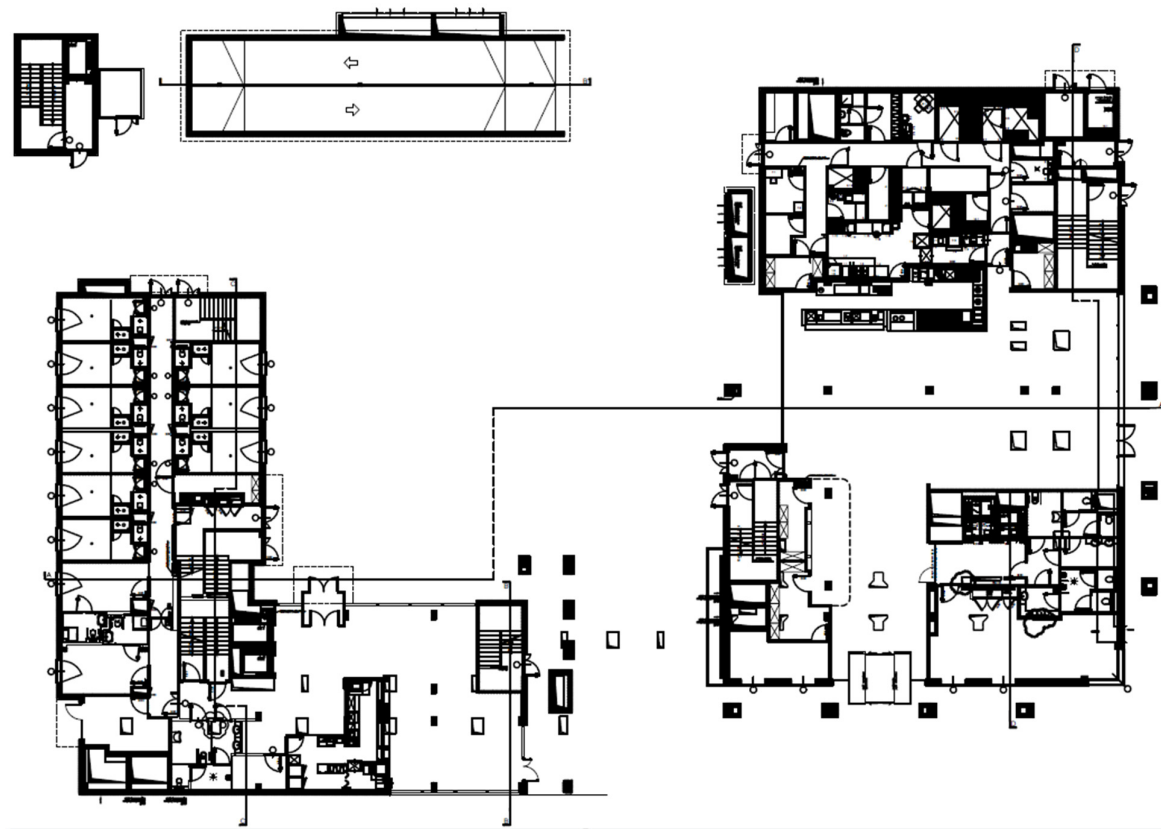


Figure 2. Ground floor plan of 2 hotels—10- and 6-storey.

These threshold ranges were adopted as practical inspection-based intervals used in technical condition assessment. They were selected to provide a consistent grading scheme for the available expert data and should not be interpreted as universal physical limits.

When the deterioration exceeds 70%, the element is considered to be in a critical state, leading to qualification for demolition or complete replacement. In addition to the numerical ranges, verbal descriptions were introduced to indicate the typical maintenance or repair actions associated with each state, ranging from routine upkeep to comprehensive renovation.

The classification was based on expert inspection of visible deterioration, material ageing, loss of functional performance, and the expected scope of repair. State 1 denotes elements with no relevant damage or only minor wear requiring routine maintenance. State 2 corresponds to limited deterioration with local defects and the need for minor repair measures. State 3 describes elements with clear degradation that still remain serviceable but require more substantial repair or refurbishment. State 4 denotes poor technical condition, with advanced wear and a high need for major intervention. The transition between states was determined by comparing the classifications from 2020 and 2025 for the same element. For the present dataset, the observed changes were monotonic, which means that an element either remained in the same state or moved to the next worse state. State 4 was treated as the terminal condition within the model, while deterioration above 70% was interpreted as a critical case requiring replacement or demolition.

Table 3 summarises the inspection results obtained in 2020 and 2025. The comparison reveals that some elements (e.g., foundations) retained their initial performance, while others exhibited notable progression to lower condition states (e.g., insulation systems, windows and doors, technical installations). This classification provides an empirical basis for constructing the frequency matrix of observed transitions. By normalising each row of

this matrix, a transition probability matrix is derived, which in turn supports probabilistic forecasting of structural deterioration and service life.

Table 3. Condition states of structural and building elements based on inspections in 2020 and 2025.

	Elements	Control 2020	Control 2025
1	foundations—reinforced concrete, slurry walls	1	1
2	structural walls—reinforced concrete	1	2
3	ceilings above the storey—reinforced concrete	2	2
4	partition walls—Silka E12 and E15	3	3
5	roof construction—reinforced concrete	3	3
6	moisture insulation (below ground)—slurry walls/waterproofing coating	3	3
7	roof covering—flat roof	3	3
8	guttering—galvanised, painted	4	4
9	moisture insulation—PVC membrane	4	4
10	thermal insulation—Styrofoam	3	4
11	windows and exterior doors—PVC (REHAU)	3	4
12	façade—ETICS—Styrofoam 15 cm, structural coating 1 mm	3	4
13	ironwork elements (balustrades, grilles)—galvanised and painted steel	2	3
14	interior doors—wooden frame, solid doors—laminate cladding	4	4
15	plaster and siding—gypsum	4	4
16	paint coatings—Seignette Soidor Blanc Albertville—RAL 9016	2	3
17	subfloors—3–5 cm thickness cement	4	4
18	floors and flooring—rooms and corridors—carpet; bathrooms—ceramic tiles (30 × 30); parking—epoxy resin SIKAFLOOR 2530 W—5 mm	4	4
19	staircases—smoothly finished concrete	2	3
20	electrical installations—standard 240 v	2	3
21	water and sewage installations—PVC	3	3
22	heating system—central heating—district	3	4
23	ventilation system—mechanical ventilation	2	3
24	external pavements/driveways—concrete blocks	3	4
25	fencing and gates—galvanised steel—painted	2	3
The number of elements qualified as:			
1	Very good	2	1
2	Medium	6	3
3	Satisfactory	9	9
4	Poor	8	12

4. Analysis of Continuous Markov Chain Results

The first step in the continuous-time modelling was the preparation of empirical data in a form suitable for probabilistic analysis. As shown in Table 3, the inspection results from 2020 and 2025 provided information about the condition of 25 structural and non-structural elements, each classified into one of four discrete deterioration states. By aligning the two inspection campaigns, it was possible to construct a count matrix **C**, in which the rows represent the initial state in 2020 and the columns correspond to the observed state in 2025. The resulting matrix is

$$\mathbf{C} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 6 \end{bmatrix} \quad (16)$$

This compact representation conveys the essential information on deterioration pathways. The first row, for instance, shows that of the two elements initially classified as **Very good**, one remained in the same state after five years while the other one deteriorated to **Medium**. The second row indicates that seven elements were originally assessed as **Medium**; one of them did not change its classification, whereas six deteriorated further to **Satisfactory**. In the third row, ten elements were in a **Satisfactory** state at the beginning of the observation period, of which half remained unchanged and half moved to the **Poor** category. Finally, the six elements already in the poorest state did not show any further change, which is consistent with the absorbing nature of this condition.

Normalising each row of the count matrix yields a discrete transition probability matrix **P**. This matrix preserves the empirical structure of the deterioration process while ensuring probabilistic consistency, namely that the entries are non-negative and each row sums to unity. For the dataset considered here the transition matrix is

$$\mathbf{P} = \begin{bmatrix} 0.5000 & 0.5000 & 0 & 0 \\ 0 & 0.1429 & 0.8571 & 0 \\ 0 & 0 & 0.5000 & 0.5000 \\ 0 & 0 & 0 & 1.0000 \end{bmatrix} \quad (17)$$

The probability matrix clearly illustrates the gradual nature of deterioration. Elements in good condition have a 50% chance of maintaining their state, but also a 50% probability of moving to the next level of wear. In contrast, elements already classified as poor remain trapped in this category with certainty, which confirms its role as an absorbing state in the chain. Such information is central to life cycle prediction, as it captures the realistic likelihood of both stability and degradation in the medium term.

While the discrete transition matrix provides an accurate description of the observed five-year changes, the aim of this study is to embed these observations into a continuous-time Markov chain framework. This requires the identification of a generator matrix **Q**, whose exponential mapping over the inspection interval reproduces the behaviour of **P**. In the MATLAB implementation, two complementary identification routes were tested. The first one was based on direct formulas linking the diagonal entries of **Q** to the observed self-transition probabilities, under the assumption of a pure-birth structure. The second relied on a least-squares optimisation procedure, in which the generator was adjusted to minimise the discrepancy between $\exp(\mathbf{Q} \Delta t)$ and the empirical matrix **P**. This second approach proved more robust, especially when the discrete data were not perfectly embeddable into a continuous-time chain.

The final generator matrix obtained from the least-squares fitting is

$$\mathbf{Q} = \begin{bmatrix} -0.1503 & 0.1503 & 0 & 0 \\ 0 & -0.3567 & 0.3567 & 0 \\ 0 & 0 & -0.1152 & 0.1152 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (18)$$

This generator satisfies the structural constraints of a continuous-time Markov chain: the off-diagonal entries are non-negative, each row sums to zero, and the last row represents the absorbing condition. In addition, the magnitudes of the negative diagonal terms reflect the average deterioration intensities observed in the data, with higher rates in the second state than in the first and third. This indicates that elements classified as medium are statistically less stable than those initially assessed as satisfactory, a feature that will have direct consequences for the long-term predictions presented in the following subsections.

The fitted rates highlight the relative instability of state 2, which exhibits the highest transition intensity ($\lambda = 0.3567$), compared to the more moderate rates for states 1 and 3. The absorbing nature of state 4 is explicitly represented by the zero row in the generator. Despite some mismatch between the exponential mapping $\exp(\mathbf{Q} \Delta t)$ and the empirical matrix $\mathbf{P}$, the fitted generator reproduces the main deterioration trends and ensures full compliance with the mathematical constraints of a CTMC.

Figure 3 compares the trajectories obtained from the discrete-time model and the continuous-time simulation with a constant generator. The results are closely aligned, confirming that the continuous formulation correctly interpolates the five-year transitions. Over the 40-year horizon, the probability mass gradually concentrates in the absorbing state 4, while the intermediate states diminish to negligible values. The trajectories also show that state 2 has only a transient role, disappearing rapidly within the first decade, whereas state 3 persists somewhat longer before also converging to zero.

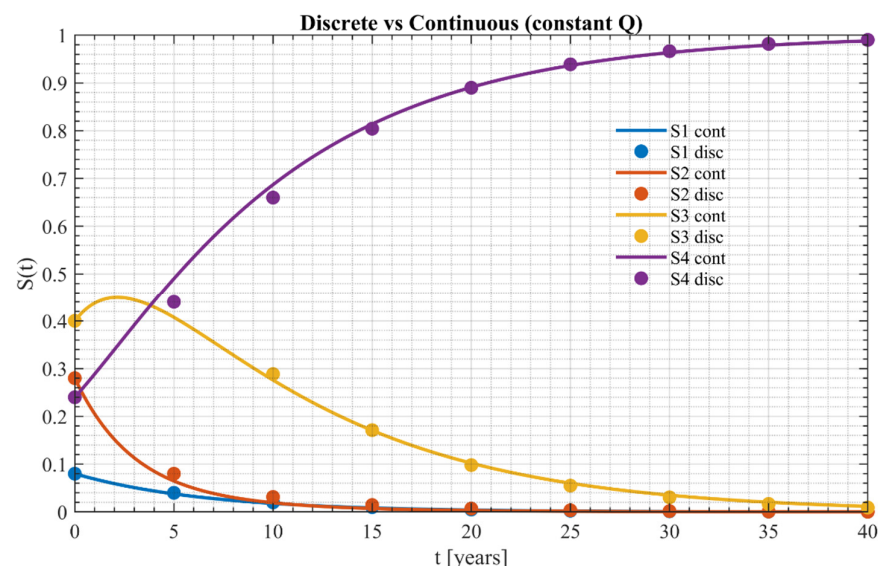


Figure 3. Constant generator results: discrete-time and continuous-time models.

Figure 4 presents the outcomes of the non-homogeneous model with time-varying transition rates, implemented through logistic growth of the intensities. Both the local exponential approximation and the ODE solver yield indistinguishable results, which validates the numerical procedures adopted in the MATLAB code. The dynamic adjustment of rates accelerates the transfer of probability mass into the absorbing state, yet the overall shape of the trajectories remains consistent with the homogeneous case. This demonstrates that the

model is robust with respect to reasonable assumptions regarding the non-stationarity of deterioration processes.

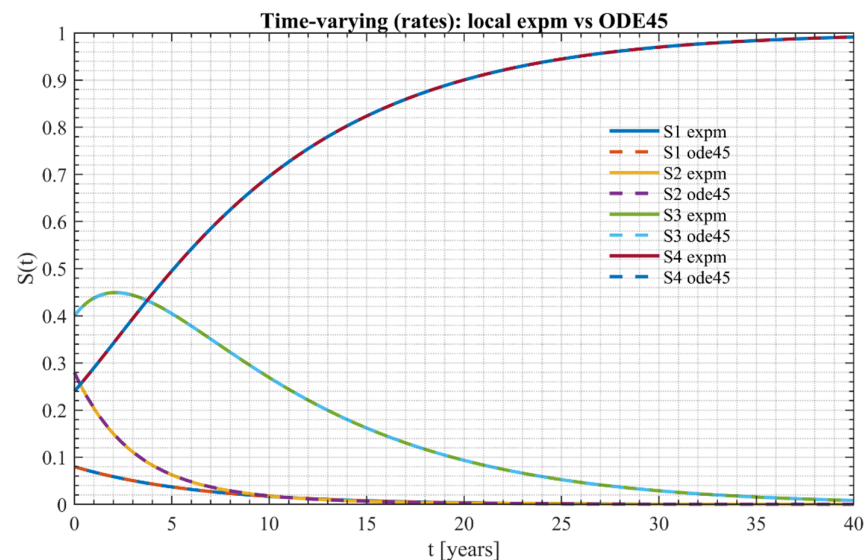


Figure 4. State probabilities for time-varying rates: local exponential versus ODE45 solver.

Finally, Figure 5 shows the comparison between the classical model and its fractional extension with order $\alpha = 0.95$. The introduction of fractional dynamics produces heavier tails in the survival of intermediate states. In particular, the decay of state 3 is noticeably slower in the fractional model, which reflects the memory effects embedded in the Mittag-Leffler kernel. As a consequence, the convergence of the process towards the absorbing state is delayed, although the limiting distribution remains unchanged. This behaviour is consistent with empirical evidence from infrastructure systems, where deterioration is often subject to temporal correlations and does not follow purely exponential laws.

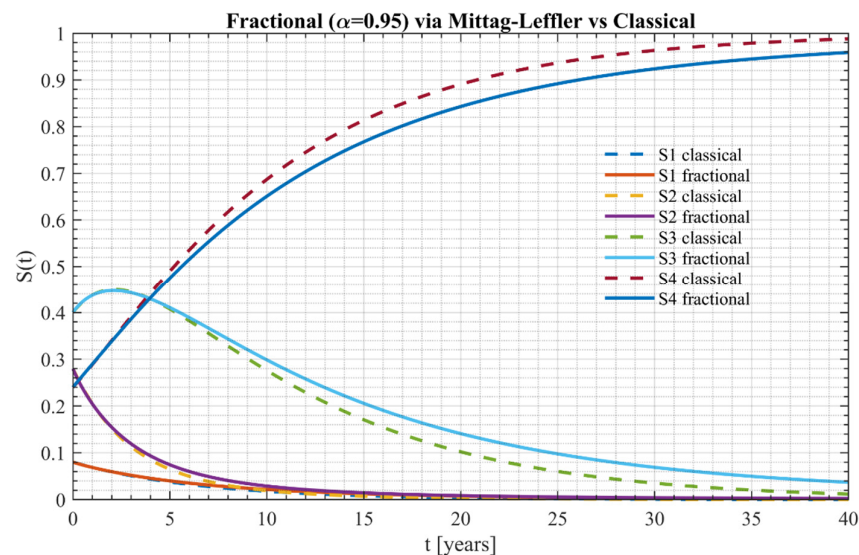


Figure 5. State probabilities for fractional versus classical Markov chain dynamics.

Taken together, these results demonstrate that the continuous-time Markov chain framework not only reproduces the empirical inspection data but also provides a flexible platform for exploring alternative deterioration dynamics. The classical, time-varying and fractional formulations represent complementary perspectives, each adding a layer of realism and interpretability to the prediction of service life.

5. Reliability and Survival Analysis

The continuous-time Markov chain model developed in the previous sections provides a probabilistic description of deterioration pathways between discrete condition states. While this formulation captures the dynamics of transitions and allows simulation of state probabilities over time, it is often desirable to summarise the results in terms more familiar to reliability and risk analysis. In infrastructure management, decision-makers frequently require estimates of the expected service life, survival probabilities over a planning horizon, and the risk of reaching a critical condition within a specified period [4].

These questions can be addressed by exploiting the absorption structure of the Markov chain. Since the poorest condition state represents a terminal situation corresponding to demolition or complete replacement, the process can be interpreted as an absorbing Markov chain, and the time to absorption becomes a natural measure of the lifetime of a building element. From the generator matrix $\mathbf{Q}$, it is possible to compute the distribution of absorption times, the corresponding survival function, and the mean time to failure (MTTF). In particular, the survival function $S(t)$ expresses the probability that the element has not yet entered the absorbing state by time t , thereby providing a direct link between Markovian deterioration modelling and classical reliability concepts [27].

This reliability-oriented extension of the continuous Markov model offers two main advantages. Firstly, it provides a compact representation of the long-term behaviour of the system, avoiding the need to track the full probability vector over time. Secondly, it allows the results to be integrated with well-established reliability frameworks, making it easier to compare Markov-based predictions with empirical survival data and with alternative stochastic models of deterioration. In the following subsections, we introduce the relevant mathematical background and illustrate its implementation in MATLAB, leading to survival curves and mean lifetime estimates for the structural elements under consideration.

5.1. Survival and Hazard Functions

Let $\mathbf{p}(t) \in \mathbb{R}^{1 \times m}$ denote the row probability vector of the continuous-time Markov chain at time t , and let A denote the set of transient (non-absorbing) states. The survival function is defined as the probability that the system has not yet reached the absorbing state by time t . It is obtained by summing the probabilities of all transient states [28,29]:

$$S(t) = \sum_{i \in A} p_i(t). \quad (19)$$

In the present model, state 4 represents absorption; therefore, the survival probability reduces to

$$S(t) = p_1(t) + p_2(t) + p_3(t). \quad (20)$$

The derivative of the survival function yields the probability density function of the absorption time:

$$f(t) = -\frac{d}{dt}S(t). \quad (21)$$

The ratio between the density and the survival function defines the hazard function, which expresses the instantaneous risk of failure conditional on survival up to time t :

$$h(t) = \frac{f(t)}{S(t)} = -\frac{d}{dt} \ln S(t). \quad (22)$$

5.2. Mean Time to Absorption

A central reliability measure is the mean time to absorption (MTTA), which quantifies the expected lifetime of the system before reaching the critical state. For a generator matrix partitioned into transient and absorbing blocks, the fundamental matrix $\mathbf{N} \in \mathbb{R}^{m_T \times m_T}$.

$$\mathbf{N} = (-\mathbf{Q}_T)^{-1} \quad (23)$$

captures the expected sojourn times in each transient state, where $\mathbf{Q}_T \in \mathbb{R}^{m_T \times m_T}$ is the restriction of the generator $\mathbf{Q} \in \mathbb{R}^{m \times m}$ to the transient part. Multiplying the initial distribution restricted to transient states by this matrix and summing the entries yields the expected lifetime [30,31].

Here, $\mathbf{Q}_T$ is assumed to be the subgenerator of the transient states in an absorbing continuous-time Markov chain. Under this assumption, eventual absorption occurs with probability one, all eigenvalues of $\mathbf{Q}_T$ have negative real parts, and therefore $-\mathbf{Q}_T$ is invertible. If these conditions are not satisfied, the fundamental matrix in Equation (21) is not defined in this form.

$$MTTA = \mathbf{p}_0^{(T)} \mathbf{N} \mathbf{I}, \quad (24)$$

where $\mathbf{I} \in \mathbb{R}^{m_T \times 1}$ is a column vector of ones.

5.3. Numerical Results

This section computes the state probabilities $\mathbf{p}(t)$, the survival function $S(t)$, the density of the absorption time $f(t)$, the hazard $h(t)$, and the mean time to absorption MTTA. For the homogeneous model the evolution is obtained from Equations (3) and (4) using the *expm* routine in MATLAB, and $S(t)$ is the sum over transient states. For the time-varying model two independent solvers are employed. The first one uses local propagation with *expm* applied to the current $\mathbf{Q}(t)$ on the time grid. The second one solves the Kolmogorov equation with *ode45* using linear interpolation of $\mathbf{Q}(t)$ [32].

In the fractional model, the derivative of order $\alpha \in (0, 1]$ introduces memory into the state evolution. This means that the current deterioration rate depends not only on the present condition state, but also on the previous history of the process. As a result, the fractional formulation can produce slower long-term decay and heavier survival tails than the classical memoryless Markov model. For $\alpha = 1$, the model reduces to the standard first-order case.

For the fractional model with $\alpha = 0.95$ we compute the solution in two independent ways. First, we use a series expansion. Second, we use an in-house implementation of the Mittag-Leffler function. Survival and hazard are obtained from the standard fractional relations, and we begin the evaluation at a very small positive time to avoid the singular behaviour at the origin that occurs for orders below one.

In the present study, the value $\alpha = 0.95$ was chosen as a representative near-classical case. This choice allows us to illustrate the influence of memory, while keeping the model close to the standard benchmark. A full calibration of α would require a larger inspection database and a dedicated inverse identification procedure. For the same reason, a broader sensitivity study with respect to α is beyond the scope of the present paper and is left for future work.

The numerical outcomes are summarised in Table 4 and visualised in Figure 6 for survival and in Figure 7 for hazards. In the homogeneous case the MTTA equals 8.1365 years from the closed form and 8.1021 years from numerical integration, which confirms internal consistency. Allowing rates to vary in time shortens the mean time to 7.8034 years. The fractional model produces 9.7342 years when computed using the series and 9.7340 years when computed using the direct Mittag-Leffler evaluation. The difference between the

two routes is 0.0002 years, which corresponds to about 0.002 percent, hence the two implementations are practically identical. At the fifty-year horizon the survival levels are 3.904×10^{-3} for the homogeneous model, 2.378×10^{-3} for the time-varying model, and 2.488×10^{-2} for the fractional model. The heavier fractional tail visible in Figure 6 explains the larger mean time produced by $\alpha = 0.95$.

Table 4. MTTA and survival at 50 years for homogeneous, time-varying and fractional CTMC models.

Model	MTTA Closed Form [Years]	MTTA Numeric [Years]	S (50 Years)
Homogeneous CTMC	8.1365	8.1021	3.904×10^{-3}
Time-varying rates (ODE)	n.a.	7.8034	2.378×10^{-3}
Fractional $\alpha = 0.95$ (series)	n.a.	9.7342	2.488×10^{-2}
Fractional $\alpha = 0.95$ (Mittag-Leffler)	n.a.	9.7340	2.488×10^{-2}

Note: n.a. = not applicable.

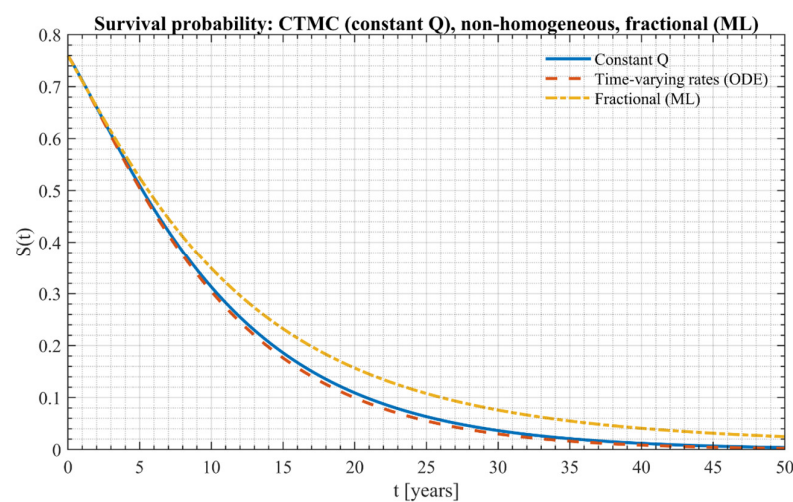


Figure 6. Survival functions for the homogeneous, time-varying, and fractional CTMC models.

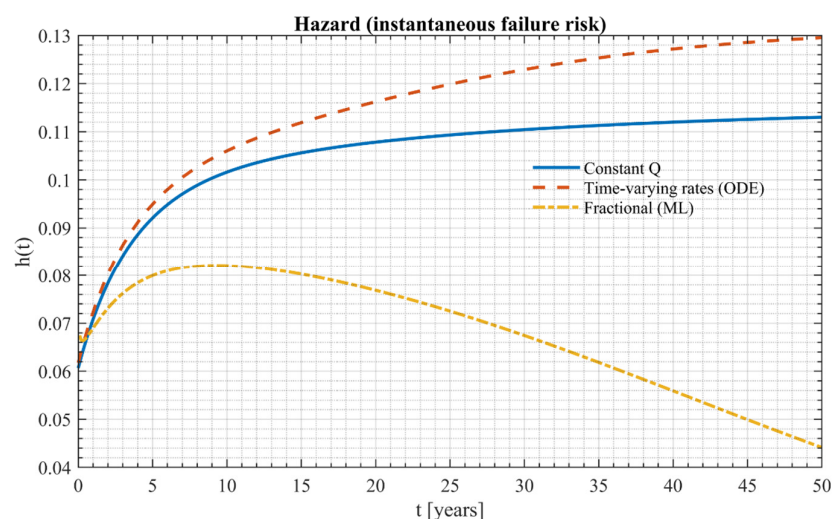


Figure 7. Hazard functions for homogeneous, time-varying, and fractional CTMC models.

To illustrate the role of the fractional order, we also examined several representative values of α below one. The parameter α controls the strength of memory in the Mittag-Leffler dynamics. When $\alpha \rightarrow 1$, the fractional model approaches the classical Markov case with exponential waiting times. When α decreases, memory effects become stronger, the waiting-time tail becomes heavier, and the mean persistence in transient states increases.

As a result, long-term survival tends to increase and absorption is delayed. From an engineering point of view, values of α close to one describe near-Markov deterioration, whereas lower values represent stronger history dependence and a wider spread of possible service life trajectories. A full calibration of α from inspection data remains a separate identification problem and is not addressed here.

The hazard plots in Figure 7 confirm that the three models imply different risk mechanisms. In the homogeneous and time-varying cases, the hazard increases monotonically over the full time window, and the time-varying curve remains above the homogeneous one. This is consistent with the classical CTMC setting, where the deterioration process remains Markovian. In the fractional case, the hazard rises rapidly at early times, reaches a broad maximum around years eight to ten, and then decreases slowly. This behaviour is not contradictory. It is a direct consequence of the Mittag-Leffler waiting-time law implied by the fractional dynamics. The heavy tail of this law means that a part of the population remains in transient states for a relatively long time, which slows the conditional risk at later times. Therefore, the non-monotone fractional hazard should be interpreted as an expected feature of the model, not as a numerical anomaly. In this sense, the effect of long memory in the fractional model is quantified here by the increase in mean time to absorption, the higher long-term survival level, and the non-monotone hazard profile relative to the classical Markov benchmarks.

The differences between the three models follow from their assumptions. The homogeneous model assumes constant transition intensities and gives a stable baseline prediction. The time-varying model allows deterioration rates to change with time, which explains the shorter mean time to absorption and lower long-term survival in the present example. The fractional model includes memory effects and therefore gives a heavier survival tail and a longer mean time to absorption.

Each model also has different practical advantages. The homogeneous model is simple and easy to calibrate, so it is useful as a reference tool when data are limited. The time-varying model is more flexible and is suitable when deterioration is expected to accelerate. The fractional model is useful for history-dependent deterioration, but its parameters are harder to identify and require careful interpretation in practical applications.

From a practical engineering perspective, the difference between monotonically increasing hazards and the non-monotonic fractional hazard reflects two complementary ways of interpreting deterioration. The classical Markov models assume memoryless progression, which leads to steadily growing risk and provides a conservative basis for scheduling inspections and interventions. The fractional model, on the other hand, incorporates long-memory effects and heavy-tailed survival, which can be interpreted as early acceleration followed by a phase of relative stabilisation. This behaviour is often observed in real assets, where initial deterioration is rapid but subsequent progression may slow down once the most vulnerable components have failed or been repaired.

Engineers responsible for asset management should not treat the two hazard patterns as mutually exclusive, but rather as alternative scenarios. The monotone curves can be used as a lower-bound, conservative benchmark for planning maintenance, while the fractional hazard provides a more optimistic but still mathematically consistent scenario that highlights the possibility of extended service life under heavy-tailed deterioration. In practice, the choice of which curve to “trust” should depend on the quality of available inspection data and on the risk appetite of decision-makers. If data are scarce, the homogeneous model ensures a safe planning margin. If evidence of long-memory behaviour is supported by field observations, then fractional dynamics may provide a better match to reality.

In summary, the fractional hazard does not contradict the classical results but complements them by broadening the range of plausible deterioration pathways. For maintenance

engineers this means that both formulations should be considered jointly. The classical models give conservative triggers for intervention, while the fractional formulation explains why some structures may remain functional much longer than predicted by memoryless models.

The proposed continuous-time framework should be viewed in relation to other prediction approaches already used in service life analysis. A discrete-time Markov chain is simpler and remains fully adequate when inspections are performed at fixed intervals and predictions are only needed at the same time grid. This was shown in our earlier study, where discrete-time and fuzzy Markov models provided a transparent description of condition-state evolution under limited data [14]. In contrast, the present continuous-time formulation allows interpolation between inspection dates, direct computation of survival and hazard as continuous functions of time, and comparison of constant, time-varying, and fractional deterioration mechanisms within one framework. Physical degradation models may offer a more direct link to material behaviour, but they usually require richer information on loads, environment, and damage evolution than is available in the present inspection dataset. For this reason, the main advantage of the proposed approach is not universal superiority in predictive error, but greater flexibility and better suitability for sparse inspection-based asset data.

6. Conclusions

A continuous-time Markov chain framework was developed for service life prediction of building elements. Three complementary variants were used. The homogeneous model relies on a constant generator and matrix exponentials. The non-homogeneous model allows transition rates to change in time and is solved with local exponentials and an ODE solver. The fractional model introduces memory effects through the Mittag-Leffler function and produces heavy-tailed survival.

This study brings several original elements. It combines classical, time-varying, and fractional formulations in one workflow that starts from inspection counts and ends with reliability measures. Two independent numerical schemes for the fractional case were implemented and their consistency confirmed the stability of the approach. Empirical inspection data were converted into a generator matrix that respects probabilistic constraints and supports long-term forecasts.

The results indicate that time-varying intensities accelerate deterioration relative to the homogeneous case. Fractional dynamics delay absorption and lead to declining hazard at long horizons, which is consistent with deterioration processes observed in practice. This contrast highlights the value of considering more than one modelling approach. The classical formulation provides a conservative benchmark for planning interventions, while the fractional variant explains why some structures may remain in service significantly longer than predicted by memoryless models.

The proposed approach is suitable for building components that can be periodically inspected and consistently assigned to discrete condition states. In this sense, it may be applied to components made of different materials, including reinforced concrete, steel, polymers, timber, and finishing materials, provided that a common inspection-based rating scheme is available. At the same time, the model operates at the condition-state level rather than at the material-mechanism level. Therefore, it is useful for service life assessment and maintenance planning, but it does not replace material-specific physical models when deterioration is governed by mechanisms such as corrosion, fatigue, creep, or chemical attack.

Future research should focus on three specific extensions of the proposed framework. First, the predicted state probabilities, survival functions, and hazard rates can be coupled

with life cycle cost and risk measures to support optimal inspection, repair, and replacement decisions. Second, the present model treats components independently, while real buildings often contain interacting elements, so multi-component formulations with dependence between deterioration processes should be developed. Third, transition intensities should be updated using condition monitoring and newly collected inspection data, for example through Bayesian or sequential updating schemes, to improve prediction accuracy over time.

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