

Chapter 1

Introduction

1.1 What Is a Stochastic Process?

A stochastic process is a probability model that describes the evolution of a system evolving randomly in time. If we observe the system at a set of discrete times, say at the end of every day or every hour, we get a discrete-time stochastic process. On the other hand, if we observe the system continuously at all times, we get a continuous-time stochastic process. We begin with examples of the discrete- and continuous-time stochastic processes.

1.2 Discrete-Time Stochastic Processes

Consider a system that evolves randomly in time. Suppose this system is observed at times $n = 0, 1, 2, 3, \dots$. Let X_n be the (random) state of the system at time n . The sequence of random variables $\{X_0, X_1, X_2, \dots\}$ is called a (discrete-time) *stochastic process* and is written as $\{X_n, n \geq 0\}$. Let S be the set of values that X_n can take for any n . Then S is called the *state space* of the stochastic process $\{X_n, n \geq 0\}$. We illustrate this with several examples below.

Example 1.1. (Examples of Discrete-Time Stochastic Processes).

- (a) Let X_n be the temperature (in degrees Fahrenheit) recorded at Raleigh-Durham Airport at 12:00 noon on the n th day. The state space of the stochastic process $\{X_n, n \geq 0\}$ can be taken to be $S = (-50, 150)$. Note that this implies that the temperature never goes below -50° or over 150°F . In theory, the temperature could go all the way down to absolute zero and all the way up to infinity!
- (b) Let X_n be the outcome of the n th toss of a six-sided die. The state space of $\{X_n, n \geq 0\}$ is $\{1, 2, 3, 4, 5, 6\}$.
- (c) Let X_n be the inflation rate in the United States at the end of the n th week. Theoretically, the inflation rate can take any real value, positive or negative. Hence the state space of $\{X_n, n \geq 0\}$ can be taken to be $(-\infty, \infty)$.

- (d) Let X_n be the Dow Jones index at the end of the n th trading day. The state space of the stochastic process $\{X_n, n \geq 0\}$ can be taken to be $[0, \infty)$.
- (e) Let X_n be the number of claims submitted to an insurance company during the n th week. The state space of $\{X_n, n \geq 0\}$ is $\{0, 1, 2, 3, \dots\}$.
- (f) Let X_n be the inventory of cars at a dealership at the end of the n th working day. The state space of the stochastic process $\{X_n, n \geq 0\}$ is $\{0, 1, 2, 3, \dots\}$.
- (g) Let X_n be the number of PCs sold at an outlet store during the n th week. The state space of the stochastic process $\{X_n, n \geq 0\}$ is the same as that in (f).
- (h) Let X_n be the number of reported accidents in Chapel Hill during the n th day. The state space of the stochastic process $\{X_n, n \geq 0\}$ is the same as that in (f). ■

1.3 Continuous-Time Stochastic Processes

Now consider a randomly evolving system that is observed continuously at all times $t \geq 0$, with $X(t)$ being the state of the system at time t . The set of states that the system can be in at any given time is called the *state space* of the system and is denoted by S . The process $\{X(t), t \geq 0\}$ is called a *continuous-time stochastic process* with state space S . We illustrate this with a few examples.

Example 1.2. (Examples of Continuous-Time Stochastic Processes).

- (a) Suppose a machine can be in two states, up or down. Let $X(t)$ be the state of the machine at time t . Then $\{X(t), t \geq 0\}$ is a continuous-time stochastic process with state space {up, down}.
- (b) A personal computer (PC) can execute many processes (computer programs) simultaneously. You can see them by invoking the task manager on your PC. Let $X(t)$ be the number of processes running on such a PC at time t . Then $\{X(t), t \geq 0\}$ is a continuous-time stochastic process with state space $\{0, 1, 2, \dots, K\}$, where K is the maximum number of jobs that can be handled by the PC at one time.
- (c) Let $X(t)$ be the number of customers that enter a bookstore during time $[0, t]$. Then $\{X(t), t \geq 0\}$ is a continuous-time stochastic process with state space $\{0, 1, 2, \dots\}$.
- (d) Let $X(t)$ be the number of emails that are in your inbox at time t . Then $\{X(t), t \geq 0\}$ is a continuous-time stochastic process with state space $\{0, 1, 2, \dots\}$.
- (e) Let $X(t)$ be the temperature at a given location at time t . Then $\{X(t), t \geq 0\}$ is a continuous-time stochastic process with state space $(-150, 130)$. Note that this implies that the temperature never goes below -150° or above 130° Fahrenheit.
- (f) Let $X(t)$ be the value of a stock at time t . Then $\{X(t), t \geq 0\}$ is a continuous-time stochastic process with state space $[0, \infty)$. In practice, the stock values are discrete, being integer multiples of .01. ■

1.4 What Do We Do with a Stochastic Process?

In this book, we shall develop tools to study systems evolving randomly in time and described by a stochastic process $\{X_n, n \geq 0\}$ or $\{X(t), t \geq 0\}$. What do we mean by “study” a system? The answer to this question depends on the system that we are interested in. Hence we illustrate the idea with two examples.

Example 1.3. (Studying Stochastic Processes).

- (a) Consider the stochastic process in Example 1.1(a). The “study” of this system may involve predicting the temperature on day 10; that is, predicting X_{10} . However, predicting X_{10} is itself ambiguous: it may mean predicting the expected value or the cdf of X_{10} . Another “study” may involve predicting how long the temperature will remain above 90° Fahrenheit assuming the current temperature is above 90° . This involves computing the mean or distribution of the random time T when the temperature first dips below 90° .
- (b) Consider the system described by the stochastic process of Example 1.1(e). The study of this system may involve computing the mean of the total number of claims submitted to the company during the first 10 weeks; i.e., $E(X_1 + X_2 + \cdots + X_{10})$. We may be interested in knowing if there is a long-term average weekly rate of claim submissions. This involves checking if $E(X_n)/n$ goes to any limit as n becomes large.
- (c) Consider the system described by the stochastic process of Example 1.1(f). Suppose it costs \$20 per day to keep a car on the lot. The study of the system may involve computing the total expected cost over the first 3 days; i.e., $E(20X_1 + 20X_2 + 20X_3)$. We may also be interested in long-run average daily inventory cost, quantified by the limit of $E(20(X_1 + X_2 + \cdots + X_n)/n)$ as n becomes large. ■

Example 1.4. (Studying Stochastic Processes).

- (a) Consider the two-state model of a machine as described in Example 1.2(a). The typical quantities of interest are:
 1. The probability that the machine is up at time t , which can be mathematically expressed as $P(X(t) = \text{up})$.
 2. Let $W(t)$ be the total amount of time the machine is up during the interval $[0, t]$. We are interested in the expected duration of the time the machine is up, up to time t , namely $E(W(t))$.
 3. The long-run fraction of the time the machine is up, defined as

$$\lim_{t \rightarrow \infty} \frac{E(W(t))}{t}.$$

4. Suppose it costs \$ c per unit time to repair the machine. What is the expected total repair cost up to time t ? What is the long-run repair cost per unit time?

- (b) Consider the computer system of Example 1.2(b). The typical quantities of interest are
1. the probability that the computer system is idle at time t ;
 2. the expected number of jobs in the system at time t , viz. $E(X(t))$; and
 3. the expected time it takes to go from idle to full. ■

The examples above show a variety of questions that can arise in the study of a system being modeled by a stochastic process $\{X_n, n \geq 0\}$ or $\{X(t), t \geq 0\}$. We urge the reader to think up plausible questions that may arise in studying the other systems mentioned in Examples 1.1 and 1.2.

In order to answer the kinds of questions mentioned above, it is clear that we must be able to compute the joint distribution of $(X_1, X_2, \dots, X_n)$ for any n in the discrete-time case or $(X(t_1), X(t_2), \dots, X(t_n))$ for any $0 \leq t_1 \leq t_2 \leq \dots \leq t_n$ in the continuous-time case. Then we can use the standard probability theory (see Appendix) to answer these questions. How do we compute this joint distribution? We need more information about the stochastic process before it can be done. If the stochastic process has a simple (yet rich enough to capture the reality) structure, then the computation is easy. Over the years, researchers have developed special classes of stochastic processes that can be used to describe a wide variety of useful systems and for which it is easy to do probabilistic computations. The two important classes are discrete-time Markov chains and continuous-time Markov chains. We begin with discrete-time Markov chains in the next chapter.