

Chapter 3

Poisson Processes

In the previous chapter, we studied a discrete-time stochastic process $\{X_n, n \geq 0\}$ on finite state space with Markov property at times $n = 0, 1, 2, \dots$. Now we would like to study a continuous-time stochastic process $\{X(t), t \geq 0\}$ on a finite state space with Markov property at each time $t \geq 0$. We shall call such a process continuous-time Markov Chain (CTMC). We shall see in the next chapter that a finite-state CTMC spends an exponentially distributed amount of time in a given state before jumping out of it. Thus exponential distributions play an important role in CTMCs. In addition, the Poisson distribution and Poisson processes also form the foundation of many CTMC models. Hence we study these topics in this chapter.

3.1 Exponential Random Variables

Consider a nonnegative random variable with parameter $\lambda > 0$ with the following pdf:

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0. \quad (3.1)$$

The corresponding cdf can be computed using Equation (B.3) as

$$F(x) = 1 - e^{-\lambda x}, \quad x \geq 0. \quad (3.2)$$

Note that if the random variable X has units of time, the parameter λ has units of time^{-1} . From Equation (3.2), we see that

$$P(X > x) = e^{-\lambda x}, \quad x \geq 0.$$

Definition 3.1. (Exponential Distribution). The pdf given by Equation (3.1) is called the exponential density. The cdf given by Equation (3.2) is called the

exponential distribution. A random variable with the cdf given in Equation (3.2) is called an exponential random variable. All three are denoted by $\text{Exp}(\lambda)$.

We leave it to the reader to verify that

$$E(X) = \frac{1}{\lambda} \quad (3.3)$$

and

$$\text{Var}(X) = \frac{1}{\lambda^2}. \quad (3.4)$$

Example 3.1. (Time to Failure). Suppose a new machine is put into operation at time zero. Its lifetime is known to be an $\text{Exp}(\lambda)$ random variable with $\lambda = .1/\text{hour}$. What are the mean and variance of the lifetime of the machine?

Using Equation (3.3), we see that the mean lifetime is $1/\lambda = 10$ hours. Similarly, Equation (3.4) yields the variance as $1/\lambda^2 = 100$ hours².

What is the probability that the machine will give trouble-free service continuously for 1 day?

To use consistent units, we use 24 hours instead of 1 day. We compute the required probability as

$$P(X > 24) = e^{-(.1)(24)} = e^{-2.4} = .0907.$$

Suppose the machine has not failed by the end of the first day. What is the probability that it will give trouble-free service for the whole of the next day?

The required probability is given by

$$\begin{aligned} P(X > 48 | X > 24) &= \frac{P(X > 48; X > 24)}{P(X > 24)} \\ &= \frac{P(X > 48)}{P(X > 24)} \\ &= \frac{e^{-(.1)(48)}}{e^{-(.1)(24)}} \\ &= e^{-(.1)(24)} \\ &= .0907. \end{aligned}$$

But this is the same as the earlier answer. Thus, given that the machine has not failed by day 1, it is as good as new! ■

The property discovered in the example above is called the *memoryless property* and is one of the most important properties of the exponential random variable. We define it formally below.

Definition 3.2. (Memoryless Property). A random variable X on $[0, \infty)$ is said to have the memoryless property if

$$P(X > t + s | X > s) = P(X > t), \quad s, t \geq 0. \quad (3.5)$$

The unique feature of an exponential distribution is that it is the only continuous nonnegative random variable with the memoryless property, as stated in the next theorem.

Theorem 3.1. (Memoryless Property of $\text{Exp}(\lambda)$). *Let X be a continuous random variable taking values in $[0, \infty)$. It has the memoryless property if and only if it is an $\text{Exp}(\lambda)$ random variable for some $\lambda > 0$.*

Proof. Suppose $X \sim \text{Exp}(\lambda)$. Then, for all $s, t \geq 0$,

$$\begin{aligned} P(X > s + t | X > s) &= \frac{P(X > s + t; X > s)}{P(X > s)} \\ &= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} \\ &= e^{-\lambda t} \\ &= P(X > t). \end{aligned}$$

Thus X has the memoryless property. Next, suppose X is a continuous random variable with pdf $f(\cdot)$. Define

$$G(x) = \int_x^\infty f(x)dx = P(X > x).$$

Assume that

$$G(0) = 1. \quad (3.6)$$

Suppose X has the memoryless property. Then

$$\begin{aligned} G(t) &= P(X > t) \\ &= P(X > t + s | X > s) \\ &= \frac{P(X > t + s, X > s)}{P(X > s)} \\ &= \frac{P(X > t + s)}{P(X > s)} \\ &= \frac{G(t + s)}{G(s)}. \end{aligned}$$

Hence we must have

$$G(t + s) = G(t)G(s), \quad s, t \geq 0.$$

Thus

$$\frac{G(t+s) - G(t)}{s} = -\frac{G(t)(1 - G(s))}{s}. \quad (3.7)$$

We have

$$\lim_{s \rightarrow 0} \frac{1 - G(s)}{s} = f(0) = \lambda \quad (\text{say}).$$

Taking limits on both sides in Equation (3.7), we get

$$G'(t) = -\lambda G(t).$$

Hence,

$$G(t) = e^{-\lambda t}, \quad t \geq 0.$$

This implies that $X \sim \text{Exp}(\lambda)$. ■

Next consider a random variable with parameters $k = 1, 2, 3, \dots$ and $\lambda > 0$ taking values in $[0, \infty)$ with the following pdf:

$$f(x) = \lambda e^{-\lambda x} \frac{(\lambda x)^{k-1}}{(k-1)!}, \quad x \geq 0. \quad (3.8)$$

Computing F from f of the equation above by using Equation (B.3) is a tedious exercise in integration by parts, and we omit the details. The final expression is

$$F(x) = 1 - \sum_{r=0}^{k-1} e^{-\lambda x} \frac{(\lambda x)^r}{r!}, \quad x \geq 0. \quad (3.9)$$

If the random variable X has units of time, the parameter λ has units of time^{-1} . From Equation (3.9), we see that

$$\mathbf{P}(X > x) = e^{-\lambda x} \sum_{r=0}^{k-1} \frac{(\lambda x)^r}{r!}, \quad x \geq 0.$$

Definition 3.3. (Erlang Distribution). The pdf given by Equation (3.8) is called the Erlang density. The cdf given by Equation (3.9) is called the Erlang distribution. A random variable with cdf given in Equation (3.9) is called an Erlang random variable. All three are denoted by $\text{Erl}(k, \lambda)$.

We leave it to the reader to verify that

$$\mathbf{E}(X) = \frac{k}{\lambda} \quad (3.10)$$

and

$$\text{Var}(X) = \frac{k}{\lambda^2}. \quad (3.11)$$

Example 3.2. (Time to Failure). We shall redo Example 3.1 under the assumption that the lifetime of the machine is an $\text{Erl}(k, \lambda)$ random variable with parameters $k = 2$ and $\lambda = .2/\text{hr}$.

Using Equation (3.10), we see that the mean lifetime is $2/\lambda = 10$ hours. Similarly, Equation (3.11) yields the variance as $2/\lambda^2 = 50$ hours².

From Equation (3.9), we have

$$\begin{aligned} \text{P}(\text{no failure in the first 24 hours}) &= \text{P}(X > 24) \\ &= e^{-(.2)(24)}(1 + (.2)(24)) \\ &= .0477 \end{aligned}$$

and

$$\begin{aligned} \text{P}(\text{no failure in the second 24 hours} \mid \text{no failure in the first 24 hours}) \\ &= \text{P}(X > 48 \mid X > 24) \\ &= \frac{\text{P}(X > 48)}{\text{P}(X > 24)} \\ &= \frac{e^{-(.2)(48)}(1 + (.2)(48))}{e^{-(.2)(24)}(1 + (.2)(24))} \\ &= .0151. \end{aligned}$$

The second probability is lower than the first, indicating that the machine deteriorates with age. This is what we would expect. ■

The next theorem shows that Erlang random variables appear naturally as the sum of iid exponential random variables.

Theorem 3.2. (Sums of Exponentials). *Suppose $\{X_i, i = 1, 2, \dots, n\}$ are iid $\text{Exp}(\lambda)$ random variables, and let*

$$Z_n = X_1 + X_2 + \dots + X_n.$$

Then Z_n is an $\text{Erl}(n, \lambda)$ random variable.

Proof. Note that $\text{Erl}(1, \lambda)$ is the same as $\text{Exp}(\lambda)$. Hence the result is true for $n = 1$. We shall show that it holds for a general n by induction. So suppose the result is true for $n = k$. We shall show that it holds for $n = k + 1$. Using

$$Z_{k+1} = Z_k + X_{k+1}$$

in Equation (C.12), we get

$$f_{Z_{k+1}}(z) = \int_{-\infty}^{\infty} f_{X_{k+1}}(z - x) f_{Z_k}(x) dx$$

$$\begin{aligned}
&= \int_0^z \lambda e^{-\lambda(z-x)} \lambda e^{-\lambda x} \frac{(\lambda x)^{k-1}}{(k-1)!} dx \\
&= \lambda^2 e^{-\lambda z} \int_0^z \frac{(\lambda x)^{k-1}}{(k-1)!} dx \\
&= \lambda e^{-\lambda z} \frac{(\lambda z)^k}{k!}.
\end{aligned}$$

The last expression can be recognized as the pdf of an $\text{Erl}(k+1, \lambda)$ random variable. Hence the result follows by induction. ■

Next we study the minimum of independent exponential random variables. Let X_i be an $\text{Exp}(\lambda_i)$ random variable ($1 \leq i \leq k$), and suppose $X_1, X_2, \dots, X_k$ are independent. Let

$$X = \min\{X_1, X_2, \dots, X_k\}. \quad (3.12)$$

We can think of X_i as the time when an event of type i occurs. Then X is the time when the first of these k events occurs. The main result is given in the next theorem.

Theorem 3.3. (Minimum of Exponentials). X of Equation (3.12) is an $\text{Exp}(\lambda)$ random variable, where

$$\lambda = \sum_{i=1}^k \lambda_i. \quad (3.13)$$

Proof. If $X_i > x$ for all $1 \leq i \leq k$, then $X > x$ and vice versa. Using this and the independence of X_i , $1 \leq i \leq k$, we get

$$\begin{aligned}
P(X > x) &= P(\min\{X_1, X_2, \dots, X_k\} > x) \\
&= P(X_1 > x, X_2 > x, \dots, X_k > x) \\
&= P(X_1 > x)P(X_2 > x) \cdots P(X_k > x) \\
&= e^{-\lambda_1 x} e^{-\lambda_2 x} \cdots e^{-\lambda_k x} \\
&= e^{-\lambda x}.
\end{aligned} \quad (3.14)$$

This proves that X is an $\text{Exp}(\lambda)$ random variable. ■

Next, let X be as in Equation (3.12), and define $Z = i$ if $X_i = X$; i.e., if event i is the first of the k events to occur. Note that X_i 's are continuous random variables, and hence the probability that two or more of them will be equal to each other is zero. Thus there is no ambiguity in defining Z . The next theorem gives the joint distribution of Z and X .

Theorem 3.4. (Distribution of (Z, X)). Z and X are independent random variables with

$$P(Z = i; X > x) = P(Z = i)P(X > x) = \frac{\lambda_i}{\lambda} e^{-\lambda x}, \quad 1 \leq i \leq k, \quad x \geq 0. \quad (3.15)$$

Proof. We have

$$\begin{aligned}
 P(Z = i; X > x) &= \int_x^\infty P(Z = i; X > x | X_i = y) \lambda_i e^{-\lambda_i y} dy \\
 &= \int_x^\infty P(X_j > y, j \neq i | X_i = y) \lambda_i e^{-\lambda_i y} dy \\
 &= \int_x^\infty \prod_{j \neq i} e^{-\lambda_j y} \lambda_i e^{-\lambda_i y} dy \\
 &= \lambda_i \int_x^\infty e^{-\lambda y} dy \\
 &= \frac{\lambda_i}{\lambda} e^{-\lambda x}.
 \end{aligned}$$

Setting $x = 0$ in the above and using the fact that X is a continuous random variable, we get

$$P(Z = i) = P(Z = i; X > 0) = P(Z = i; X \geq 0) = \frac{\lambda_i}{\lambda}. \quad (3.16)$$

Hence we see that

$$P(Z = i; X > x) = P(Z = i)P(X > x),$$

proving the independence of Z and X . ■

It is counterintuitive that Z and X are independent random variables because it implies that the time until the occurrence of the first of the k events is independent of which of the k events occurs first! In yet other terms, the conditional distribution of X , given that $Z = i$, is $\text{Exp}(\lambda)$ and not $\text{Exp}(\lambda_i)$ as we might have (incorrectly) guessed.

Example 3.3. (Boy or a Girl?). A maternity ward at a hospital currently has seven pregnant women waiting to give birth. Three of them are expected to give birth to boys, and the remaining four are expected to give birth to girls. From prior experience, the hospital staff knows that a mother spends on average 6 hours in the hospital before delivering a boy and 5 hours before delivering a girl. Assume that these times are independent and exponentially distributed. What is the probability that the first baby born is a boy and is born in the next hour?

Let X_i be the time to delivery of the i th expectant mother. Assume mothers 1, 2, and 3 are expecting boys and mothers 4, 5, 6, and 7 are expecting girls. Then

our assumption implies that $\{X_i, 1 \leq i \leq 7\}$ are independent exponential random variables with parameters $1/6, 1/6, 1/6, 1/5, 1/5, 1/5$, and $1/5$, respectively. Let B be the time when the first boy is born, and G be the time when the first girl is born. Then Theorem 3.3 implies

$$B = \min\{X_1, X_2, X_3\} \sim \text{Exp}(3/6),$$

$$G = \min\{X_4, X_5, X_6, X_7\} \sim \text{Exp}(4/5).$$

Also, B and G are independent. Then the probability that the first baby is a boy is given by

$$P(B < G) = \frac{3/6}{3/6 + 4/5} = \frac{5}{13} = 0.3846.$$

The time of the first birth is

$$\min\{B, G\} \sim \text{Exp}(3/6 + 4/5) = \text{Exp}(1.3).$$

Thus the probability that the first baby is born within the hour is given by

$$P(\min\{B, G\} \leq 1) = 1 - \exp(-1.3) = 0.7275.$$

Using Equation (3.15), we get the probability that the first baby born is a boy and is born in the next hour as $0.3846 * 0.7275 = 0.2798$. ■

In the next section, we study another random variable that is quite useful in building stochastic models.

3.2 Poisson Random Variables

Suppose we conduct n independent trials of an experiment. Each trial is successful with probability p or unsuccessful with probability $1 - p$. Let X be the number of successes among the n trials. The state space of X is $\{0, 1, 2, \dots, n\}$. The random variable X is called a binomial random variable with parameters n and p and is denoted as $\text{Bin}(n, p)$. The pmf of X is given by

$$p_k = P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad 0 \leq k \leq n. \quad (3.17)$$

The pmf above is called the binomial distribution. Computing this distribution for large values of n is difficult since the term $\binom{n}{k}$ becomes very large, while the term $p^k (1 - p)^{n-k}$ becomes very small. To avoid such difficulties, it is instructive to look at the limit of the binomial distribution as $n \rightarrow \infty$ and $p \rightarrow 0$ in such a way that np approaches a fixed number, say $\lambda \in (0, \infty)$. We shall denote this limiting region as

D to avoid clutter in the following derivation. We have

$$\begin{aligned}
 & \lim_D \binom{n}{k} p^k (1-p)^{n-k} \\
 &= \lim_D \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \\
 &= \lim_D \frac{(np)^k}{k!} \left(\prod_{r=0}^k \left(\frac{n-r}{n} \right) \right) \left(1 - \frac{np}{n} \right)^{n-k} \\
 &= \lim_D \frac{(np)^k}{k!} \left(\lim_D \prod_{r=0}^k \left(\frac{n-r}{n} \right) \right) \lim_D \left(1 - \frac{np}{n} \right)^n \lim_D \left(1 - \frac{np}{n} \right)^{-k} \\
 &= \left(\frac{\lambda^k}{k!} \right) (1) (e^{-\lambda}) (1) \\
 &= e^{-\lambda} \frac{\lambda^k}{k!}.
 \end{aligned} \tag{3.18}$$

Here we have used

$$\lim_D \left(1 - \frac{np}{n} \right)^n = \lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n} \right)^n = e^{-\lambda}.$$

Thus, in the limit, the $\text{Bin}(n, p)$ random variable approaches a random variable with state space $S = \{0, 1, 2, \dots\}$ and pmf

$$p_k = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k \in S. \tag{3.19}$$

The pmf above plays an important part in probability models and hence has been given the special name Poisson distribution.

Definition 3.4. (Poisson Distribution). A random variable with pmf given by Equation (3.19) is called a Poisson random variable with parameter λ and is denoted by $P(\lambda)$. The pmf given in Equation (3.19) is called a Poisson distribution with parameter λ and is also denoted by $P(\lambda)$.

We see that Equation (3.19) defines a proper pmf since

$$\begin{aligned}
 \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \\
 &= e^{-\lambda} e^{\lambda} = 1.
 \end{aligned}$$

We leave it to the reader to verify that, if X is a $P(\lambda)$,

$$E(X) = \lambda \quad (3.20)$$

and

$$\text{Var}(X) = \lambda. \quad (3.21)$$

Example 3.4. (Counting Accidents). Suppose accidents occur one at a time at a dangerous traffic intersection during a 24-hour day. We divide the day into 1440 minute-long intervals and assume that there can be only 0 or 1 accidents during each 1-minute interval. Let E_k be the event that there is an accident during the k th minute-long interval, $1 \leq k \leq 1440$. Suppose that the event E_k 's are mutually independent and that $P(E_k) = .001$. Compute the probability that there are exactly k accidents during one day.

Let X be the number of accidents during a 24-hour period. Using the assumptions above, we see that X is a $\text{Bin}(1440, .001)$ random variable. Hence

$$P(X = k) = \binom{1440}{k} (.001)^k (.999)^{1440-k}, \quad 0 \leq k \leq 1440.$$

This is numerically difficult to compute even for moderate values of k . Hence we approximate X as a $P(1440 * .0001) = P(1.440)$ random variable. This yields

$$P(X = k) = e^{-1.440} \frac{1.440^k}{k!}, \quad k \geq 0.$$

For example, for $k = 0$, the binomial formula produces a value .2368, while the Poisson formula produces .2369. ■

Example 3.5. (Service Facility). Customers arrive at a service facility one at a time. Suppose that the total number of arrivals during a 1-hour period is a Poisson random variable with parameter 8. Compute the probability that at least three customers arrive during 1 hour.

Let X be the number of arrivals during 1 hour. Then we are given that X is a $P(8)$ random variable. The required probability is given by

$$\begin{aligned} P(X \geq 3) &= 1 - P(X = 0) - P(X = 1) - P(X = 2) \\ &= 1 - e^{-8}(1 + 8 + 64/2) \\ &= 1 - 41e^{-8} = .9862. \end{aligned}$$

What are the mean and variance of the number of arrivals in a 1-hour period? Using Equations (3.20) and (3.21), we see that the mean and variance are $\lambda = 8$. ■

Next we study the sums of independent Poisson random variables. The main result is given in the next theorem.

Theorem 3.5. (Sums of Poissons). Suppose $\{X_i, i = 1, 2, \dots, n\}$ are independent random variables with $X_i \sim P(\lambda_i)$, $1 \leq i \leq n$. Let

$$Z_n = X_1 + X_2 + \cdots + X_n.$$

Then Z_n is a $P(\lambda)$ random variable, where

$$\lambda = \sum_{i=1}^n \lambda_i.$$

Proof. We shall show the result for $n = 2$. The general result then follows by induction. Using Equation (C.11), we get

$$\begin{aligned} p_Z(k) &= \sum_{k_2=0}^k e^{-\lambda_1} \frac{\lambda_1^{k-k_2}}{(k-k_2)!} e^{-\lambda_2} \frac{\lambda_2^{k_2}}{k_2!} \\ &= e^{-\lambda_1-\lambda_2} \frac{1}{k!} \sum_{k_2=0}^k \frac{k!}{(k-k_2)!k_2!} \lambda_1^{k-k_2} \lambda_2^{k_2} \\ &= e^{-\lambda_1-\lambda_2} \frac{(\lambda_1 + \lambda_2)^k}{k!}. \end{aligned} \quad (3.22)$$

The last equality follows from the binomial identity

$$\sum_{r=0}^k \frac{k!}{(k-r)!r!} \lambda_1^{k-r} \lambda_2^r = (\lambda_1 + \lambda_2)^k.$$

This completes the proof. ■

Since adding two Poisson random variables generates another Poisson random variable, it is natural to ask: can we construct a reverse process? That is, can we produce two Poisson random variables starting from one Poisson random variable? The answer is affirmative. The reverse process is called thinning and can be visualized as follows. Let X be a $P(\lambda)$ random variable representing the random number of events. An event may or may not be registered. Let $p \in [0, 1]$ be a fixed number representing the probability that any given event is registered, independent of everything else. Let X_1 be the number of registered events and X_2 be the number of unregistered events. Now, from the definition of the random variables, it is clear that $X_1 \sim \text{Bin}(X, p)$ and $X_2 = X - X_1$. Hence

$$\begin{aligned} P(X_1 = i, X_2 = j) &= P(X_1 = i, X_2 = j | X = i + j) P(X = i + j) \\ &= \binom{i+j}{i} p^i (1-p)^j e^{-\lambda} \frac{\lambda^{i+j}}{(i+j)!} \\ &= e^{-\lambda p} \frac{(\lambda p)^i}{i!} e^{-\lambda(1-p)} \frac{(\lambda(1-p))^j}{j!}, \quad i \geq 0, j \geq 0. \end{aligned}$$

From the joint distribution above, we can compute the following marginal distributions:

$$\begin{aligned} P(X_1 = i) &= e^{-\lambda p} \frac{(\lambda p)^i}{i!}, \quad i \geq 0, \\ P(X_2 = j) &= e^{-\lambda(1-p)} \frac{(\lambda(1-p))^j}{j!}, \quad j \geq 0. \end{aligned}$$

Hence we have

$$P(X_1 = i, X_2 = j) = P(X_1 = i)P(X_2 = j). \quad (3.23)$$

Thus X_1 is $P(\lambda p)$ and X_2 is $P(\lambda(1-p))$, and they are mutually independent. The independence is quite surprising!

Using the properties of the exponential and Poisson distributions studied above, we shall build our first continuous-time stochastic process in the next section.

3.3 Poisson Processes

Frequently we encounter systems whose state transitions are triggered by streams of events that occur one at a time, for example, arrivals to a queueing system, shocks to an engineering system, earthquakes in a geological system, biological stimuli in a neural system, accidents in a given city, claims on an insurance company, demands on an inventory system, failures in a manufacturing system, etc. The modeling of these systems becomes more tractable if we assume that the successive inter-event times are iid exponential random variables. For reasons that will become clear shortly, such a stream of events is given a special name: *Poisson process*. We define it formally below.

Let S_n be the occurrence time of the n th event. Assume $S_0 = 0$, and define

$$T_n = S_n - S_{n-1}, \quad n \geq 1.$$

Thus T_n is the time between the occurrence of the n th and the $(n-1)$ st event. Let $N(t)$ be the total number of events that occur during the interval $(0, t]$. Thus an event at 0 is not counted, but an event at t , if any, is counted in $N(t)$. One can formally define it as

$$N(t) = \max\{n \geq 0 : S_n \leq t\}, \quad t \geq 0. \quad (3.24)$$

Definition 3.5. (Poisson Process). The stochastic process $\{N(t), t \geq 0\}$, where $N(t)$ is as defined by Equation (3.24), is called a Poisson process with rate λ (denoted by $PP(\lambda)$) if $\{T_n, n \geq 1\}$ is a sequence of iid $\text{Exp}(\lambda)$ random variables.

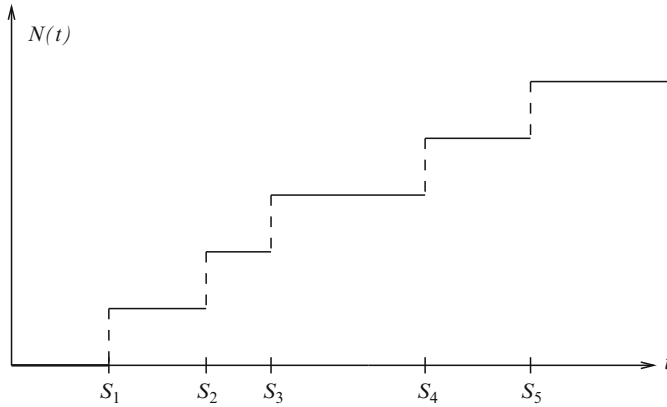


Fig. 3.1 A typical sample path of a Poisson process.

A typical sample path of a Poisson process is shown in Figure 3.1. The next theorem gives the pmf of $N(t)$ for a given t . It also justifies the name Poisson process for this stochastic process.

Theorem 3.6. (Poisson Process). *Let $\{N(t), t \geq 0\}$ be a $PP(\lambda)$. For a given t , $N(t)$ is a Poisson random variable with parameter λt ; i.e.,*

$$P(N(t) = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}, \quad k \geq 0. \quad (3.25)$$

Proof. We shall first compute $P(N(t) \geq k)$ and then obtain $P(N(t) = k)$ by using

$$P(N(t) = k) = P(N(t) \geq k) - P(N(t) \geq k + 1). \quad (3.26)$$

Suppose $N(t) \geq k$. Then the k th event in the Poisson process must have occurred at or before t ; i.e., $S_k \leq t$. On the other hand, suppose $S_k \leq t$. Then the k th event has occurred at or before t . Hence there must be at least k events up to t ; i.e., $N(t) \geq k$. Thus the events $\{N(t) \geq k\}$ and $\{S_k \leq t\}$ are identical and must have the same probability. However, S_k is a sum of k iid $\text{Exp}(\lambda)$ random variables. Hence, from Theorem 3.2, it is an $\text{Erl}(k, \lambda)$ random variable. Using Equation (3.9), we get

$$P(N(t) \geq k) = P(S_k \leq t) = 1 - \sum_{r=0}^{k-1} e^{-\lambda t} \frac{(\lambda t)^r}{r!}.$$

Substituting in (3.26), we get

$$\begin{aligned}
 P(N(t) = k) &= P(N(t) \geq k) - P(N(t) \geq k + 1) \\
 &= 1 - \sum_{r=0}^{k-1} e^{-\lambda t} \frac{(\lambda t)^r}{r!} - \left(1 - \sum_{r=0}^k e^{-\lambda t} \frac{(\lambda t)^r}{r!} \right) \\
 &= e^{-\lambda t} \frac{(\lambda t)^k}{k!}.
 \end{aligned}$$

This completes the proof. ■

As an immediate consequence of the theorem above, we get

$$E(N(t)) = \lambda t, \quad \text{Var}(N(t)) = \lambda t.$$

Thus the expected number of events up to t increases linearly in t with rate λ . This justifies calling λ the rate of the Poisson process. The next theorem gives an important property of the Poisson process.

Theorem 3.7. (Markov Property). *Let $\{N(t), t \geq 0\}$ be a $PP(\lambda)$. It has the Markov property at each time t ; that is,*

$$P(N(t + s) = k | N(s) = j, N(u), 0 \leq u \leq s) = P(N(t + s) = k | N(s) = j). \quad (3.27)$$

Idea of Proof. This can be seen as follows. Fix a $t \geq 0$, and suppose $N(t) = k$ is given. Then S_{k+1} is the time of occurrence of the first event after t . Using the (strong) memoryless property (see Conceptual Problem 3.11) of the exponential random variables, we can show that the time until occurrence of this next event, namely $S_{k+1} - t$, is exponentially distributed with parameter λ and is independent of the history of the Poisson process up to t . Thus the time until the occurrence of the next event in a Poisson process is always $\text{Exp}(\lambda)$, regardless of the occurrence of events so far. Equation (3.27) follows as a consequence. ■

The result above can be stated in another way: the distribution of the increment $N(t + s) - N(s)$ over any interval $(s, t + s]$ is independent of the events occurring outside this interval. In particular, the increments over non-overlapping intervals are independent. In fact, the argument in the proof above shows that the number of events on any time interval $(s, s + t]$ is given by

$$N(t + s) - N(s) \sim P(\lambda t).$$

Thus we have

$$P(N(t + s) = k | N(s) = j) = e^{-\lambda t} \frac{(\lambda t)^{k-j}}{(k-j)!}, \quad 0 \leq j \leq k.$$

Thus the distribution of the increment $N(t + s) - N(s)$ over any interval $(s, t + s]$ depends only on t , the length of the interval. We say that the increments are stationary. We say that the Poisson process has *stationary and independent increments*. We shall see this property again when we study Brownian motion in Chapter 7.

Example 3.6. Let $N(t)$ be the number of births in a hospital over the interval $(0, t]$. Suppose $\{N(t), t \geq 0\}$ is a Poisson process with rate 10 per day.

1. What are the mean and variance of the number of births in an 8-hour shift?

Using days as the time unit, we see that we want to compute the mean and variance of $N(1/3)$. Now $N(1/3) \sim P(10/3)$. Hence we get

$$E(N(1/3)) = 10/3, \quad \text{Var}(N(1/3)) = 10/3.$$

2. What is the probability that there are no births from noon to 1 p.m.?

Using 1 hour = $1/24$ day, we see that the number of births in one hour window is given by $N(1/24) \sim P(10/24)$. Hence the required probability is given by

$$P(N(1/24) = 0) = \exp(-10/24) = 0.6592.$$

3. What is the probability that there are three births from 8 a.m. to 12 noon and four from 12 noon to 5 p.m.?

Assuming that 8 a.m. is $t = 0$, the desired probability is given by

$$\begin{aligned} & P(N(4/24) - N(0) = 3; N(8/24) - N(4/24) = 4) \\ &= P(N(4/24) - N(0) = 3)P(N(8/24) - N(4/24) = 4) \\ &\quad \text{(independence of increments)} \\ &= P(N(4/24) = 3)P(N(4/24) = 4) \\ &\quad \text{(stationarity of increments)} \\ &= e^{-10/6} \frac{(10/6)^3}{3!} e^{-10/6} \frac{(10/6)^4}{4!} \\ &= .0088. \end{aligned}$$

This calculation illustrates the usefulness of the stationary and independent increments property of a Poisson process. ■

Poisson processes and their variations and generalizations have been extensively studied, and we study a few of them in the rest of this chapter. However, the reader can safely skip these sections since the material on Poisson processes presented so far is sufficient for the purposes of modeling encountered in the rest of this book.

3.4 Superposition of Poisson Processes

Superposition of stochastic processes is the equivalent of addition of random variables. It is a natural operation in the real world. For example, suppose $N_1(t)$ is the number of male arrivals at a store up to time t and $N_2(t)$ is the number of female arrivals at that store up to time t . Then $N(t) = N_1(t) + N_2(t)$ is the total number of arrivals at the store up to time t . The process $\{N(t), t \geq 0\}$ is called the superposition of $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$. The next theorem gives the probabilistic structure of the superposed process $\{N(t), t \geq 0\}$.

Theorem 3.8. (Superposition). *Let $\{N_1(t), t \geq 0\}$ be a $\text{PP}(\lambda_1)$ and $\{N_2(t), t \geq 0\}$ be a $\text{PP}(\lambda_2)$. Suppose the two processes are independent of each other. Let $\{N(t), t \geq 0\}$ be the superposition of $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$. Then $\{N(t), t \geq 0\}$ is a $\text{PP}(\lambda_1 + \lambda_2)$.*

Proof. Let T_i be the time of the first event in the $\{N_i(t), t \geq 0\}$ process ($i = 1, 2$). From the definition of the Poisson process, we see that T_i is an $\text{Exp}(\lambda_i)$ random variable for $i = 1, 2$. The independence of the two Poisson processes implies that T_1 and T_2 are independent. Now let T be the time of the first event in the superposed process. Since the superposed process counts events in both the processes, it follows that $T = \min\{T_1, T_2\}$. From Theorem 3.3 it follows that T is an $\text{Exp}(\lambda_1 + \lambda_2)$ random variable.

Now consider the $\{N_i(t), t \geq 0\}$ processes from time T onward. From the property of the exponential random variables, it follows that $\{N_1(t + T), t \geq 0\}$ and $\{N_2(t + T), t \geq 0\}$ are independent Poisson processes with parameters λ_1 and λ_2 , respectively. Thus we can repeat the argument above to see that the time until the first event in the process $\{N(t + T), t \geq 0\}$ is again an $\text{Exp}(\lambda_1 + \lambda_2)$ random variable, and it is independent of the past up to time T . Thus it follows that the inter-event times in the superposed process are iid $\text{Exp}(\lambda_1 + \lambda_2)$ random variables. Hence $\{N(t), t \geq 0\}$ is a $\text{PP}(\lambda_1 + \lambda_2)$. ■

From Theorem 3.6 we see that, for a fixed t , $N_i(t) \sim \text{P}(\lambda_i t)$ ($i = 1, 2$) are independent. Hence it follows from Theorem 3.8 that $N(t) \sim \text{P}(\lambda_1 t + \lambda_2 t)$. We could have derived this directly from Theorem 3.5.

Example 3.7. (Superposition). Let $B(t)$ be the number of boys born in a hospital over $(0, t]$. Similarly, let $G(t)$ be the number of girls born at this hospital over $(0, t]$. Assume that $\{B(t), t \geq 0\}$ is a Poisson process with rate 10 per day, while $\{G(t), t \geq 0\}$ is an independent Poisson process with rate 9 per day. What are the mean and variance of the number of births in a week?

Let $N(t)$ be the number of births during $(0, t]$. Then $N(t) = B(t) + G(t)$. From Theorem 3.8, we see that $\{N(t), t \geq 0\}$ is a Poisson process with rate 19 per day. The desired answer is given by

$$E(N(7)) = \text{Var}(N(7)) = 19 * 7 = 133. \quad \blacksquare$$

3.5 Thinning of a Poisson Process

Let $\{N(t), t \geq 0\}$ be a $PP(\lambda)$ and let $0 \leq p \leq 1$ be a given real number. Suppose every event in the Poisson process is registered with probability p , independent of everything else. Let $R(t)$ be the number of registered events over $(0, t]$. The process $\{R(t), t \geq 0\}$ is called a thinning of $\{N(t), t \geq 0\}$. The next theorem describes the process $\{R(t), t \geq 0\}$.

Theorem 3.9. (Thinning). *The thinned process $\{R(t), t \geq 0\}$ as constructed above is a $PP(\lambda p)$.*

Proof. Let $\{T_n, n \geq 1\}$ be inter-event times in the $\{N(t), t \geq 0\}$ process. From the definition of a Poisson process, it follows that $\{T_n, n \geq 1\}$ is a sequence of iid $\text{Exp}(\lambda)$ random variables. Let K be the first event in the N process that is registered. Then K is a geometric random variable with probability mass function

$$p_k = \mathbf{P}(K = k) = (1 - p)^{k-1} p, \quad k = 1, 2, 3, \dots$$

Let T be the time of the first event in the R process. We have

$$T = \sum_{i=1}^K T_i.$$

We see that, for a fixed k , if $K = k$, $T = T_1 + T_2 + \dots + T_k$. From Theorem 3.2, we see that, given $K = k$, T is an $\text{Erl}(k, \lambda)$ random variable with density

$$f_{T|K=k}(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{k-1}}{(k-1)!}, \quad t \geq 0.$$

Now we can use conditioning to compute the probability density $f_T(\cdot)$ of T as follows:

$$\begin{aligned} f_T(t) &= \sum_{k=1}^{\infty} f_{T|K=k}(t) \mathbf{P}(K = k) \\ &= \sum_{k=1}^{\infty} \lambda e^{-\lambda t} \frac{(\lambda t)^{k-1}}{(k-1)!} (1-p)^{k-1} p \\ &= \lambda p e^{-\lambda t} \sum_{k=1}^{\infty} \frac{(\lambda t)^{k-1}}{(k-1)!} (1-p)^{k-1} \\ &= \lambda p e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda(1-p)t)^k}{k!} \\ &= \lambda p e^{-\lambda t} e^{\lambda(1-p)t} \\ &= \lambda p e^{-\lambda p t}. \end{aligned}$$

Thus T is an $\text{Exp}(\lambda p)$ random variable. Since each event is registered independently, it follows that the successive inter-event times in the R process are iid $\text{Exp}(\lambda p)$ random variables. This proves that $\{R(t), t \geq 0\}$ is a $\text{PP}(\lambda p)$ as desired. ■

We state one more important property of the thinning process. We have defined $R(t)$ as the number of registered events up to time t . That means $N(t) - R(t)$ is the number of unregistered events up to time t . We say the $N(t)$ is split into $R(t)$ and $N(t) - R(t)$.

Theorem 3.10. (Splitting). $\{N(t) - R(t), t \geq 0\}$ is a $\text{PP}(\lambda(1 - p))$ and is independent of $\{R(t), t \geq 0\}$.

Proof. Each event of the $\{N(t), t \geq 0\}$ process is counted in the $\{N(t) - R(t), t \geq 0\}$ process with probability $1 - p$. Hence it follows from Theorem 3.9 that $\{N(t) - R(t), t \geq 0\}$ is a $\text{PP}(\lambda(1 - p))$. Proving independence is more involved. We shall satisfy ourselves with the independence of the marginal distributions. This follows directly from Equation (3.23), which implies that $R(t)$ and $N(t) - R(t)$ are independent for a fixed t . ■

Example 3.8. (Thinning). Let $N(t)$ be the number of births in a hospital over $(0, t]$. Suppose $\{N(t), t \geq 0\}$ is a Poisson process with rate 16 per day. The hospital experience shows that 48% of the births are boys and 52% are girls. What is the probability that we see six boys and eight girls born on a given day?

We assume that each baby is a boy with probability .48 and a girl with probability .52, independent of everything else. Let $B(t)$ be the number of boys born in a hospital over $(0, t]$. Similarly, let $G(t)$ be the number of girls born at this hospital over $(0, t]$. Then, from Theorem 3.10, we see that $\{B(t), t \geq 0\}$ is a $\text{PP}(16 \cdot .48) = \text{PP}(7.68)$ and is independent of $\{G(t), t \geq 0\}$, which is a $\text{PP}(16 \cdot .52) = \text{PP}(8.32)$. Thus $B(1) \sim P(7.68)$ is independent of $G(1) \sim P(8.32)$. Hence the desired probability is given by

$$P(B(1) = 6, G(1) = 8) = e^{-7.68} \frac{7.68^6}{6!} e^{-8.32} \frac{8.32^8}{8!} = 0.0183. \quad \blacksquare$$

3.6 Compound Poisson Processes

From Definition 3.5 and Figure 3.1, we see that events occur one at a time in a Poisson process. Now we define a process that is useful in modeling a situation when multiple events can occur at the same time. For example, customers arrive in batches at a restaurant. Thus, if we define $X(t)$ to be the total number of arrivals at this restaurant over $(0, t]$, it cannot be adequately modeled as a Poisson process. We introduce the compound Poisson processes as a model of such a situation.

Definition 3.6. (Compound Poisson Process). Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ . Let $\{C_n, n \geq 1\}$ be a sequence of iid random variables with mean τ and second moment s^2 . Suppose it is also independent of the Poisson process. Let

$$C(t) = \sum_{n=1}^{N(t)} C_n, \quad t \geq 0. \quad (3.28)$$

The stochastic process $\{C(t), t \geq 0\}$ is called a *compound Poisson process (CPP)*.

The next theorem gives the expressions for the mean and variance of $C(t)$. The computation of the distribution of $C(t)$ is hard.

Theorem 3.11. (Mean and Variance of a CPP). *Let $\{C(t), t \geq 0\}$ be a CPP. Then*

$$\mathbb{E}(C(t)) = \tau \lambda t, \quad (3.29)$$

$$\text{Var}(C(t)) = s^2 \lambda t. \quad (3.30)$$

Proof. From Section 3.3 we see that $N(t)$ is a Poisson random variable with parameter λt , and hence,

$$\mathbb{E}(N(t)) = \text{Var}(N(t)) = \lambda t.$$

Note that $C(t)$ is a random sum of random variables. Using Equation (D.9), we get

$$\begin{aligned} \mathbb{E}(C(t)) &= \mathbb{E}\left(\sum_{n=1}^{N(t)} C_n\right) \\ &= \mathbb{E}(C_1) \mathbb{E}(N(t)) \\ &= \tau \lambda t. \end{aligned}$$

From Equation (D.10), we get

$$\begin{aligned} \text{Var}(C(t)) &= \text{Var}\left(\sum_{n=1}^{N(t)} C_n\right) \\ &= \mathbb{E}(N(t)) \text{Var}(C_1) + (\mathbb{E}(C_1))^2 \text{Var}(N(t)) \\ &= \lambda t (s^2 - \tau^2 + \tau^2) \\ &= \lambda t s^2. \end{aligned}$$

This proves the theorem. ■

Equation (3.29) makes intuitive sense since λt is the expected number of batches that arrive until t and each batch brings in τ customers on average. Equation (3.30), on the other hand, is a bit surprising since one would expect the variance of C_1 to appear where s^2 appears. This is the consequence of the variance of $N(t)$. We end with a numerical example.

Example 3.9. (Restaurant Arrival Process). Consider the process of customers arriving at a restaurant. Suppose the customers arrive in parties (or batches) of variable

sizes. The successive party sizes are iid random variables and are binomially distributed with parameters $n = 10$ and $p = .4$. Thus the mean size of a party is $\tau = np = 4$ and its variance is $np(1 - p) = 2.4$. Hence the second moment is $s^2 = 2.4 + 4 * 4 = 18.4$. Note that a binomial random variable can take a value 0 with probability $(1 - p)^n$. What does it mean to get a batch arrival of size 0? The parties themselves arrive according to a Poisson process with a rate of 10 per hour. Let $C(t)$ be the total number of arrivals up to time t , assuming $C(0) = 0$. Then $\{C(t), t \geq 0\}$ is a compound Poisson process. Equation (3.29) yields

$$E(C(t)) = \tau \lambda t = (4)(10)t = 40t,$$

and Equation (3.30) yields

$$\text{Var}(C(t)) = s^2 \lambda t = (18.4)(10)t = 184t.$$

Suppose the process above is a valid model of the arrival process over the time interval from 7 p.m. to 10 p.m. Then the expected total number of arrivals over those 3 hours will be given by $40 * 3 = 120$, and its variance is $184 * 3 = 552$. ■

3.7 Problems

CONCEPTUAL PROBLEMS

- 3.1. Find the mode of a $\text{Bin}(n, p)$ distribution.
- 3.2. Find the mode of a $P(\lambda)$ distribution.
- 3.3. Find the mode of an $\text{Erl}(k, \lambda)$ density.
- 3.4. Find the median of an $\text{Exp}(\lambda)$ distribution.
- 3.5. Compute the hazard rate of an $\text{Exp}(\lambda)$ random variable.
- 3.6. Let h be the hazard rate of a random variable X with cdf F . Show that

$$1 - F(x) = \exp\left(-\int_0^x h(u)du\right).$$

This shows that the hazard rate uniquely determines the cdf.

- 3.7. Compute $E(X(X - 1))$, where X is a $P(\lambda)$ random variable.
- 3.8. Let X be an $\text{Exp}(\lambda)$ random variable. Show that

$$E(X^n) = n!/\lambda^n.$$

3.9. Let X be a $P(\lambda)$ random variable. Show that

$$E(X(X-1)(X-2)\cdots(X-k+1)) = \lambda^k, \quad k \geq 1.$$

3.10. Let $X_1 \sim P(\lambda_1)$ and $X_2 \sim P(\lambda_2)$ be two independent random variables. Show that, given $X_1 + X_2 = k$, $X_i \sim \text{Bin}(k, \lambda_i/(\lambda_1 + \lambda_2))$, $i = 1, 2$.

3.11. Suppose $T_i, i = 1, 2$, the lifetimes of machine i , are independent $\text{Exp}(\lambda_i)$ random variables. Suppose machine 1 fails before machine 2. Show that the remaining lifetime of machine 2 from the time of failure of machine 1 is an $\text{Exp}(\lambda_2)$ random variable; i.e., show that

$$P(T_2 > T_1 + t | T_2 > T_1) = e^{-\lambda_2 t}.$$

This is sometimes called the *strong memoryless property*.

3.12. Let T_1 and T_2 be two iid $\text{Exp}(\lambda)$ random variables. Using Conceptual Problem 3.11 or otherwise, show that

$$E(\max(T_1, T_2)) = \frac{1}{2\lambda} + \frac{1}{\lambda}.$$

3.13. Generalize Conceptual Problem 3.12 as follows. Let $T_i, 1 \leq i \leq k$, be k iid $\text{Exp}(\lambda)$ random variables. Define

$$T = \max(T_1, T_2, \dots, T_k).$$

Show that

$$E(T) = \frac{1}{\lambda} \sum_{i=1}^k \frac{1}{i}.$$

3.14. Let T be as in Conceptual Problem 3.13. Compute $P(T \leq t)$.

3.15. A system consists of n independent components. Let X_i be the lifetime of the i th component, and assume that $\{X_i, 1 \leq i \leq n\}$ are iid $\text{Exp}(\lambda)$. The system as a whole functions if at least k components are functioning at time T ($1 \leq k \leq n$), where T is the mission time of the system. What is the probability that the system functions? (Such a system is called a k out of n system.)

3.16. Let $\{X_n, n = 1, 2, \dots\}$ be a set of iid $\text{Exp}(\lambda)$ random variables and N be an independent $G(p)$ random variable. Show that

$$Z = X_1 + X_2 + \cdots + X_N$$

is an $\text{Exp}(\lambda p)$ random variable.

3.17. Let $\{N(t), t \geq 0\}$ be a $PP(\lambda)$. Let S_k be the time of occurrence of the k th event in this Poisson process. Show that, given $N(t) = 1$, S_1 is uniformly distributed over $[0, t]$.

3.18. Suppose customers arrive at a service station with two servers according to a $PP(\lambda)$. An arriving customer is routed to server 1 or server 2 with equal probability. What are the mean and variance of the time between two consecutive arrivals to the first server? Next suppose we route the arriving customers to the two servers alternately. Now what are the mean and variance of the time between two consecutive arrivals to the first server? Is the arrival process of the customers to server 1 a Poisson process?

3.19. Let $\{N(t), t \geq 0\}$ be a $PP(\lambda)$. Let $s, t \geq 0$. Compute the joint distribution

$$P(N(s) = i, N(s+t) = j), \quad 0 \leq i \leq j < \infty.$$

3.20. Let $\{N(t), t \geq 0\}$ be a $PP(\lambda)$. Show that

$$\text{Cov}(N(s), N(s+t)) = \lambda s, \quad s, t \geq 0.$$

3.21. A bus with infinite capacity runs on an infinite route with stops indexed $n = 1, 2, \dots$. The bus arrives empty at stop 1. When the bus arrives at stop n , each passenger on it gets off with probability p , independent of each other. Then all the waiting passengers get on. Let W_n be the number of passengers waiting at stop n , and suppose $\{W_n, n \geq 0\}$ are iid $P(\lambda)$ random variables. Let R_n be the number of passengers on the bus when it leaves stop n . What is the distribution of R_n ?

3.22. In Computational Problem 3.21, compute the expected number of passengers that get off at stop n .

3.23. Let $\{C(t), t \geq 0\}$ be a CPP with batch sizes of mean 1 and second moment 1. Show that $\{C(t), t \geq 0\}$ is in fact a $PP(\lambda)$.

3.24. Let $\{C(t), t \geq 0\}$ be a CPP of Definition 3.6. Compute $E(C(t+s) - C(s))$ and $\text{Var}(C(t+s) - C(s))$ for $s, t \geq 0$.

3.25. Let $\{C(t), t \geq 0\}$ be a CPP of Definition 3.6. Compute $\text{Cov}(C(t+s), C(s))$ for $s, t \geq 0$.

COMPUTATIONAL PROBLEMS

3.1. The lifetime of a car (in miles) is an $\text{Exp}(\lambda)$ random variable with $1/\lambda = 130,000$ miles. If the car is driven 13,000 miles a year, what is the distribution of the lifetime of the car in years?

3.2. The lifetimes of two car batteries (brands A and B) are independent exponential random variables with means 12 hours and 10 hours, respectively. What is the probability that the Brand B battery outlasts the Brand A battery?

3.3. Suppose a machine has three independent components with $\text{Exp}(1)$ lifetimes. Compute the expected lifetime of the machine if it needs all three components to function properly.

3.4. A communications satellite is controlled by two computers. It needs at least one functioning computer to operate properly. Suppose the lifetimes of the computers are iid exponential random variables with mean 5 years. Once a computer fails, it cannot be repaired. What is the cdf of the lifetime of the satellite? What is the expected lifetime of the satellite?

3.5. The run times of two computer programs, A and B, are independent random variables. The run time of program A is an exponential random variable with mean 3 minutes, while that of program B is an $\text{Erl}(2, \lambda)$ random variable with mean 3 minutes. If both programs are being executed on two identical computers, what is the probability that program B will finish first?

3.6. A spacecraft is sent to observe Jupiter. It is supposed to take pictures of Jupiter's moons and send them back to Earth. There are three critical systems involved: the camera, the batteries, and the transmission antenna. These three systems fail independently of each other. The mean lifetime of the battery system is 6 years, that of the camera is 12 years, and that of the antenna is 10 years. Assume all lifetimes are exponential random variables. The spacecraft is to reach Jupiter after 3 years to carry out this mission. What is the probability that the mission is successful?

3.7. In Computational Problem 3.6, suppose the monitoring station on Earth conducts a test run after 2 years and finds that no data are received, indicating that one of the three systems has failed. What is the probability that the camera has failed?

3.8. A car comes equipped with one spare tire. The lifetimes of the four tires at the beginning of a long-distance trip are iid exponential random variables with a mean of 5000 miles. The spare tire has an exponential lifetime with a mean of 1000 miles. Compute the expected number of miles that can be covered without having to go to a tire shop. (*Hint:* Compute the expected time until the first tire failure and then that until the second tire failure.)

3.9. A customer enters a bank and finds that all three tellers are occupied and he is the next in line. Suppose the mean transaction times are 5 minutes and are independent exponential random variables. What is the expected amount of time that he has to wait before he is served?

3.10. In Computational Problem 3.9, suppose the bank also has a teller machine outside with one person currently using it and no one waiting behind him. The processing time at the machine is exponentially distributed with mean 2 minutes. Should the customer in Computational Problem 3.9 wait at the machine or at the inside tellers if the aim is to minimize the expected waiting time until the start of service? Assume that once he decides to wait in one place, he does not change his mind.

3.11. The weight of a ketchup bottle is Erlang distributed with $k = 10$ and $\lambda = .25 \text{ Oz}^{-1}$. The bottle is declared to be underweight if the weight is under 39 Oz. What is the probability that the bottle is declared underweight?

3.12. Two vendors offer functionally identical products. The expected lifetime of both the products is 10 months. However, the distribution of the first is $\text{Exp}(\lambda)$, while that of the other is $\text{Erl}(2, \mu)$. If the aim is to maximize the probability that the lifetime of the product is greater than 8 months, which of the two products should be chosen?

3.13. The lifetime of an item is an exponential random variable with mean 5 days. We replace this item upon failure or at the end of the fourth day, whichever occurs first. Compute the expected time of replacement.

3.14. Consider the 2 out of 3 system of Conceptual Problem 3.15 with the mean lifetime of each component being 3 years. Suppose that initially all the components are functioning. We visit this system after 3 years and replace all the failed components at a cost of \$75.00 each. If the system has failed, it costs us an additional \$1000.00. Compute the expected total cost incurred at time 3.

3.15. In Computational Problem 3.9, suppose customers arrive according to a PP with rate 12 per hour. At time zero, all three tellers are occupied and there is no one waiting. What is the probability that the next arriving customer will have to wait for service?

3.16. The number of industrial accidents in a factory is adequately modeled as a PP with a rate of 1.3 per year. What is the probability that there are no accidents in one year? What is the expected time (in days) between two consecutive accidents?

3.17. A computer system is subject to failures and self-repairs. The failures occur according to a PP with rate 2 per day. The self-repairs are instantaneous. The only effect of a failure is that the work done on any program running on the machine at the time of failure is lost and the program has to be restarted from scratch. Now consider a program whose execution time is 3 hours if uninterrupted by failures. What is the probability that the program completes successfully without a restart?

3.18. Redo Computational Problem 3.17 if the program execution time (in hours) is a random variable that is uniformly distributed over (2,4).

3.19. A machine is subject to shocks arriving from two independent sources. The shocks from source 1 arrive according to a PP with rate 3 per day and those from source 2 at rate 4 per day. What are the mean and variance of the total number of shocks from both the sources over an 8-hour shift.

3.20. In Computational Problem 3.19, what is the probability that the machine is subject to a total of exactly two shocks in 1 hour?

3.21. In Computational Problem 3.19, suppose a shock from source i can cause the machine to fail with probability p_i , independent of everything else. Suppose $p_1 = .011$ and $p_2 = .005$. A failed machine is replaced instantaneously. Let $N(t)$ be the number of machine replacements over the interval $(0, t]$. Is $\{N(t), t \geq 0\}$ a Poisson process? If it is, what is its parameter?

3.22. Consider a maternity ward in a hospital. A delivery may result in one, two or three births with probabilities .9, .08, and .02, respectively. The number of deliveries forms a Poisson process with rate 10 per day. Formulate the number of births as a compound Poisson process, and compute the mean number of births during one day.

3.23. In Computational Problem 3.22, compute the probability of at least one twin being born on a given day.

3.24. In Computational Problem 3.22, suppose a twin is born during a 24-hour interval. What is the expected number of deliveries that occurred during that interval?

3.25. Suppose 45% of the cars on the road are domestic brands and 55% are foreign brands. Let $N(t)$ be the number of cars crossing at a given toll booth during $(0, t]$. Suppose $\{N(t), t \geq 0\}$ is a PP with rate 40 per minute. What is the probability that four foreign cars and five domestic cars cross the intersection during a 10-second interval?

3.26. In Computational Problem 3.25, suppose 30 foreign cars have crossed the toll booth during a 1-minute interval. What is the expected number of domestic cars that cross the intersection during this interval given this information?

3.27. Consider the toll booth of Computational Problem 3.25. Suppose the toll paid by a car at the toll booth is \$1.00 with probability .3 and \$2.00 with probability .7. What are the mean and variance of the toll collected at the toll booth during 1 hour?

3.28. Suppose the distribution of the toll depends on whether the car is domestic or foreign as follows. The domestic cars pay \$1.00 with probability .4 and \$2.00 with probability .6, while the foreign cars pay \$1.00 with probability .25 and \$2.00 with probability .75. Now compute the mean and the variance of the total toll collected at the booth over a 1-hour period.

3.29. The number of cars visiting a national park forms a PP with rate 15 per hour. Each car has k occupants with probability p_k as given below:

$$p_1 = .2, \quad p_2 = .3, \quad p_3 = .3, \quad p_4 = .1, \quad p_5 = .05, \quad p_6 = .05.$$

Compute the mean and variance of the number of visitors to the park during a 10-hour window.

3.30. Now suppose the national park of Computational Problem 3.29 charges \$4.00 per car plus \$1.00 per occupant as the entry fee. Compute the mean and variance of the total fee collected during a 10-hour window.